September 2013 Written Certification Exam

Algebra

1. Let V be a 3-dimensional $\mathbb{Q}$-vector space, and let $T : V \to V$ be a linear operator that has eigenvalues 1 and 2 but is not diagonalizable.

   (a) What are the possible rational canonical forms of T?

   (b) What are the possible Jordan canonical forms of the operator $\mathrm{Id} \otimes T : \mathbb{C} \otimes_{\mathbb{Q}} V \to \mathbb{C} \otimes_{\mathbb{Q}} V$ on the complexification?

2. Let A be an integral domain.

   (a) Define what it means for an element $\pi \in A$ to be irreducible.

   (b) Suppose that $\pi \in A$ is irreducible. Show that the polynomial ring $A[x]$ is not a PID.

   (c) Show that $A[x]$ is a PID if and only if A is a field.

3. Let V be a finite-dimensional vector space over a field k of characteristic zero, and let $\langle \cdot, \cdot \rangle : V \times V \to k$ be a skew-symmetric bilinear form.

   (a) State what it means to say that the form is *nondegenerate*.

   (b) Let $W \subseteq V$ be a subspace of such that the restriction $\langle \cdot, \cdot \rangle : |_{W \times W} : W \times W \to k$ is nondegenerate. Show that V admits an orthogonal decomposition $V = W \boxplus W^{\perp}$, where $W^{\perp} = \{x \in V : \forall w \in W, \langle x, w \rangle = 0\}$. Show also that if the bilinear form on V was nondegenerate, then so is its restriction to $W^{\perp}$.

   (c) Show that if the form is nondegenerate on V, then V is even-dimensional, and it has a basis relative to which the Gram matrix of the form is

   $$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$

   where I_n is the $n \times n$ identity matrix.

4. Let K be a field of prime characteristic p, $\mathbb{F}_p$ the finite field with p elements.

   (a) First assume that $K/\mathbb{F}_p$ is an algebraic extension. Show that for every $\alpha \in K$, there is a unique $\beta \in K$ with $\beta^p = \alpha$.

   (b) Now let K be an arbitrary field of characteristic p, and assume that L/K is a finite extension with $[L : K] = n$ and $p \nmid n$. Show that L/K is a separable extension of fields.

5. A nonabelian group G has exactly three conjugacy classes. What group is G, and why?

6. Let $n = 13 \cdot 29 = 377$, and $m \geq 3$ a square-free integer. Let L be the splitting field over $\mathbb{Q}$ of $(x^7 - m)(x^n - 1)$.

   (a) Determine the splitting field $L/\mathbb{Q}$ and its degree over $\mathbb{Q}$, justifying all steps.

   (b) Determine whether or not $\mathrm{Gal}(L/\mathbb{Q})$ is abelian.

   (c) Determine whether or not $\mathrm{Gal}(L/\mathbb{Q})$ is a solvable group, and if so, give an appropriate normal tower which demonstrates this fact. If not, be clear why the extension fails to have a solvable Galois group.

Topology

1. Let X and Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Let $X \times Y$ have the product topology. Show that $\pi(X \times Y, (x_0, y_0))$ is isomorphic to $\pi(X, x_0) \times \pi(Y, y_0)$.

2. Let M be a smooth manifold, X a continuous vector field on M (i.e., a continuous section of the tangent bundle TM). There are two reasonable definitions of what it means for X to be smooth at a point p in M:

   (a) Definition 1: Let (x, U) be a local coordinate system defined on an open neighborhood U of p; then X can be expressed in local coordinates as $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ for some real-valued functions $a^1, \ldots, a^n$ defined on U. Then X is *smooth* at p provided that each coefficient function a^i is smooth at p.

   (b) Definition 2: The vector field X is *smooth* at p if for every smooth function f defined on a neighborhood of p, the function $X(f)$ is smooth at p.

   Prove that these two definitions are equivalent.

3. Show that S^{n-1} is not a retract of $E^n = \{\, x \in \mathbf{R}^n : |x| \leq 1 \,\}$ for $n \geq 1$. Use this to prove the Brouwer Fixed-Point Theorem; that is, show that if $n \geq 1$, then any continuous map $f : E^n \to E^n$ must have a fixed point.

4. **a** Does a boundary of a parallelizable manifold have to be a parallelizable manifold? Prove your answer.

   b Does a product of two parallelizable manifolds have to be a parallelizable manifold? Prove your answer.

   c Is the Klein bottle a parallelizable manifold? How about the torus $S^1 \times S^1$? Prove your answer.

5. Let $n \geq 2$ and $B \subset S^n$ be a wedge of two circles; that is, B is a closed subset of S^n homeomorphic to a figure eight so that $B = C \cup D$ with C and D homeomorphic to S^1 and $C \cap D$ a single point. Compute $H_q(S^n \setminus B)$ for $n \geq 2$.

6. **a** Let $\phi : S^2 \to \mathbb{R}^{17}$ be a smooth map. Let ω be a closed 2-form on $\mathbb{R}^{17}$. Compute the integral $\int_{S^2} \phi^* \omega$.

   b Let $\phi : S^3 \to S^2$ and $\psi : S^2 \to S^4$ be smooth maps of oriented manifolds. Let ω be a 3-form on S^4. Compute $\int_{S^3} (\psi \circ \phi)^* \omega$.

Analysis

1. Suppose f is entire and $\lim_{z \to \infty} f(z) \in \mathbb{C}$ exists. Show that f is constant.

2. Let $(V, (\cdot, \cdot))$ be an inner product space over the field $\mathbb{F}$.

   a.) If $\mathbb{F} = \mathbb{R}$, show that vectors $x, y \in V$ are orthogonal **if and only** if

   $$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

   b.) Show that (a) is *false* for any complex ($\mathbb{F} = \mathbb{C}$) inner product space V, where x can be **any** nonzero vector in V. (Hint: y should be more imaginary than x.)

3. In each of the following, you are given a domain D and a function $f : D \to \mathbb{C}$. Determine whether f has an anti-derivative on D.

   (a) $f(z) = e^{1/z} Log(z)$ where D is the complex plane with the origin and negative real axis removed.

   (b) $f(z) = \frac{1}{z^2 - 1}$ where D consists of all points in $\mathbb{C}$ except for ± 1.

   (c) $f(z) = \exp(\frac{1}{z^2})$, where $D = \mathbb{C} - \{0\}$.

4. Consider $C[0,1]$ with the uniform norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$. Show that the linear map

   $$V : C[0,1] \to C[0,1]$$

   defined by the formula

   $$V(f)(x) = \int_0^x f(t)\, dt$$

   is a bounded linear operator with **no** eigenvalues.

5. Find the limit of each of the following sequences of integrals. Justify fully. (Here m denotes Lebesgue measure on $\mathbb{R}$.)

(a) $\lim_{n\to\infty} \int_{[0,\infty)} f_n \, dm$ where $f_n(x) = \frac{\sin(nx)}{n(1+x^2)}$

(b) $\lim_{n\to\infty} \int_{[0,\infty)} f_n \, dm$ where $f_n(x) = e^{-\frac{x}{n}} \frac{1}{1+x}$.

6. Let f, g be 2π-periodic (Lebesgue) measurable functions on $\mathbb{R}$. Let $f * g$ denote the (normalized) *convolution* function

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(x-t)\, dt.$$

a.) Show that if (their restrictions) $f, g \in L^2[-\pi, \pi]$ then $f * g(x)$ exists and is bounded on $[-\pi, \pi]$, in fact,

$$\|f * g\|_\infty = \sup_{x\in[-\pi,\pi]} |f * g(x)| \leq \frac{1}{2\pi} \|f\|_2 \|g\|_2.$$

b.) Show also that $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in \mathbb{Z}$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

is the n-th Fourier coefficient of f for $n \in \mathbb{Z}$.