June 2013 Written Certification Exam

Algebra

1. Let P be a p-Sylow subgroup of a finite group G such that for every other p-Sylow subgroup Q, we have $P \cap Q = 1$. Show that any pair P_1, P_2 of p-Sylow subgroups intersects trivially: $P_1 \cap P_2 = 1$.

2. Let k be a field, and x, y indeterminates over k.

   (a) Show that x and y are irreducible in $k[x, y]$.

   (b) Show that as rings $k[x, y]/(y - x^2)$ can never be isomorphic to $k[x, y]/(y^2 - x^2)$.

   (c) Determine the structure of the quotient ring $\mathbb{Q}[x]/(x^{12} - 1)$ by characterizing this ring as a direct product of simple (quotient) rings.

3. Let V be a finite-dimensional vector space over a field k, and let $T : V \to V$ be a linear operator whose characteristic polynomial generates the ideal $I \subseteq k[X]$ in the polynomial ring consisting of polynomials that vanish at T, i.e., $I = \{f \in k[X] : f(T) = 0\}$. Show that any linear operator $U \in \mathrm{End}_k(V)$ that commutes with T is a polynomial in T; i.e., if $UT = TU$, then there is some $p \in k[X]$ such that $U = p(T)$.

4. Let E, F, and K be fields all contained in some larger extension Ω.

   (a) Suppose that $K \subset F \subset E$. Show that E/F and F/K are algebraic extensions implies that E/K is algebraic.

   (b) Suppose that E/K is an algebraic extension of fields, but that F/K is an arbitrary extension. Show that the extension EF/F is algebraic where EF is the compositum of E and F.

5. Let k be a field, and let V and W be k-vector spaces. Let $V^* := \mathrm{Hom}_k(V, k)$ denote the dual space of V.

   (a) Define a natural map $F : V^* \otimes_k W \to \mathrm{Hom}_k(V, W)$ of vector spaces that is an isomorphism if V and W are finite-dimensional. (Be sure to show that F is well-defined. You need not prove naturality, but be sure to state what it means to say that F is natural.)

   (b) Recall that a *projection* on a finite-dimensional k-vector space V is an idempotent linear operator $P \in \mathrm{End}_k(V)$. Determine necessary and sufficient conditions on $\varphi \in V^*$ and $v \in V$ insuring that the decomposable tensor $\varphi \otimes v \in V^* \otimes_k V$ corresponds, via the linear isomorphism $F : V^* \otimes_k V \to \mathrm{End}_k(V)$ above, to a nonzero projection operator.

6. Let K/F be a finite separable extension and L the Galois closure of K in some fixed algebraic closure $\overline{F}$ of F. Let G be the Galois group $\mathrm{Gal}(L/F)$ and H the subgroup corresponding to K under the Galois correspondence.

(a) Show that there is a one-to-one correspondence between the set of embeddings $\sigma : K/F \to \overline{F}$ (that is of K into $\overline{F}$ fixing F pointwise), and the set of cosets G/H.

(b) Recall that one defines the norm from K to F as follows: For $\alpha \in K$, define $N_{K/F}(\alpha) = \prod_\sigma \sigma(\alpha)$ where the product is taken over all the embeddings $\sigma : K/F \to \overline{F}$. Show that $N_{K/F}(K) \subseteq F$.

Topology

1. Prove that the Lie bracket of two vector fields is a vector field.

2. If $1 \leq n < m$, show that no open subset of $\mathbf{R}^n$ is homeomorphic to an open subset of $\mathbf{R}^m$.

3. **a** Does there exist a manifold whose boundary is the disjoint union of two Klein bottles? Construct such a manifold or prove that it does not exist.

   b Does there exist an orientable manifold whose boundary is the disjoint union of two Klein bottles? Construct such a manifold or prove that it does not exist.

   c Does there exist a Lie group whose boundary is a torus $S^1 \times S^1$? Construct such a Lie group or prove that it does not exist.

4. Let G be a topological group; that is, G is a group equipped with a topology such that multiplication $\mu : G \times G \to G$ and inversion $\iota : G \to G$ are continuous. Show that the fundamental group $\pi(G, e)$ is abelian.

5. Prove that the wedge product of differential forms gives a well defined operation on the cohomology group of the manifold. (This operation is called the cup product of cohomology classes.)

6. Suppose that A and B are subspaces of X and that B is a deformation retract of A. Show that $H_q(X, B) \cong H_q(X, A)$ for all $q \geq 0$. (You may use the 5-lemma without proof.)

Analysis

1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is everywhere analytic (i.e., entire).

   (a) Show that the function $g(z) = f(\bar{z})$ is entire only if f is a constant function.

   (b) Show that the function $h(z) = \overline{f(\bar{z})}$ is entire.

2. Let $C[0, 1]$ denote the vector space of all continuous complex-valued functions $f : [0, 1] \to \mathbb{C}$. Show that
$$S = \{f \in C[0, 1] : f(0) = 0\}$$
is a linear subspace of $C[0, 1]$. Give $C[0, 1]$ the supremum (uniform) norm $\| \cdot \|_\infty$:
$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$
Is S a closed subspace? Why or why not?

3. Let (X, M, μ) be a measure space. Let $h : X \to [0, \infty]$ be an M-measurable function on X. Define $\lambda : \mathrm{M} \to [0, \infty]$ by

$$\lambda(E) = \int_E h \, d\mu.$$

Show that λ is a measure on (X, M).

4. Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$. If S is any nonempty subset of $\mathcal{H}$ and V the closed subspace generated by S, i.e., $V = \overline{\mathrm{span}(S)}$, show that $S^\perp = V^\perp$, i.e., their orthogonal complements are equal.

5. Let $\{a_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}$. We state two definitions of $\limsup a_n$ below. Show definition (a) implies the statement in (b). (You don't have to prove the converse.)

   (a) $\limsup a_n = \lim_{n\to\infty} (\sup\{a_k : k \geq n\})$.

   (b) $\limsup a_n$ is the largest subsequential limit of $\{a_n\}_{n=1}^\infty$. (Recall that $a \in [-\infty, \infty]$ is said to be a subsequential limit of $\{a_n\}_{n=1}^\infty$ if some subsequence $\{a_{n_k}\}_{k=1}^\infty$ satisfies $\lim_{k\to\infty} a_{n_k} = a$.)

6. Let V and W be Banach spaces. A bounded linear operator operator $A \in L(V, W)$ is said to be *bounded below* if there is a constant $C > 0$ such that

$$\|A(x)\| \geq C\|x\|, \quad \forall x \in V.$$

   a.) Show that if A is bounded below, then A is injective and has closed range.

   b.) Show that if A is bounded below then $A^{-1} : \mathrm{Range}(T) \to V$ is bounded. Thus, if A has dense range then $A^{-1} \in L(W, V)$.