September 2012 Written Certification Exam

Algebra questions

1. Let L be the splitting field of $x^{15} - 8$ over $\mathbb{Q}$, and let G be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$.

   (a) Show that G is a semidirect product of two proper subgroups K and H.

   (b) Identify the subgroups K, H as subgroups of the Galois group, i.e., in terms of intermediate fields, and determine their isomorphism types.

2. Give three equivalent conditions which characterize when an algebraic extension of fields L/K is a normal extension, and prove any two are equivalent.

3. Let F be a field of characteristic 0, $f \in F[x]$ an irreducible polynomial of degree $n \geq 1$, and K the splitting field of f over F. It should be well-known that $[K : F] \leq n!$. The point of this problem is to show $[K : F] \mid n!$. Hint: prove that there exists an injective homomorphism $\mathrm{Gal}(K/F) \to S_n$ where S_n is the symmetric group on n letters.

4. Let G be a group of order pqr, where $p < q < r$ are distinct primes. Show that G is solvable.

5. Let K be the subgroup of $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by the three elements:
   $u_1 = (1, -3, -2)$, $u_2 = (1, 3, 2)$, and $u_3 = (3, 3, 4)$. Determine the structure of the quotient G/K as a direct sum of cyclic groups.

6. Let R be a commutative ring. An R-module M is *flat* if the functor $M \otimes_R (\cdot)$ is exact. Prove that any projective R-module is flat.

Topology questions

1. Let M be a smooth manifold, and let $x^1, \ldots, x^n$ be a local coordinate system defined on an open set $U \subseteq M$. Consider the $(1,1)$-tensor field C defined on U in local coordinates by

$$C = \sum_{i=1}^{n} dx^i \otimes \frac{\partial}{\partial x^i}.$$

   Show that C is independent of the choice of local coordinates and hence defines a smooth global tensor field on M.

2. Determine the the critical points of the determinant mapping $\det : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ defined on the space of $n \times n$ matrices. [**Hint:** The determinant is multilinear as a function of the columns of a matrix.]

3. Let $S \subseteq \mathbb{R}^3$ be the surface with boundary given by

$$S = \{(x, y, z) : z = x^2 + y^2, z \leq 9\},$$

oriented by the unit normal field $N = (n_1, n_2, n_3)$ with $n_3 < 0$. Let ω be the 2-form on $\mathbb{R}^3$ given by

$$\omega = e^{z \sin y}\, dy \wedge dz + \tan^{-1}(x \sinh z)\, dz \wedge dx + 2\, dx \wedge dy.$$

Compute the integral $\int_S \omega$.

4. Suppose that a space X is the disjoint union $X = U \sqcup V$ of two open subspaces U and V.

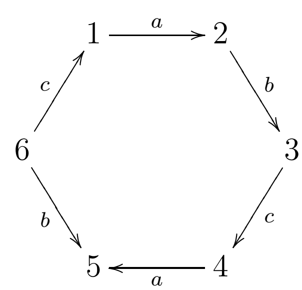
   (a) Use the Eilenberg-Steenrod axioms to prove that for any homology theory, the homology groups of X are given in terms of those of U and V by

   $$H_q(X) = H_q(U) \oplus H_q(V).$$

   (b) Why is this result easier if we take the homology theory to be singular homology?

5. Let $p : Y \to X$ be a covering map. Let Z be any connected space, and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

6. Consider the space X obtained as the quotient space of a planar hexagon and its interior by identifying boundary edges of the hexagon in pairs according to the following scheme:



Compute the homology groups of X.

Analysis questions

1. State the Hahn-Banach Theorem and use it to show that if B is a Banach space, then its dual, B^*, of bounded linear functionals separates points of B. (That is, you are asked to show that if a and b are distinct elements of B, then there is a $\phi \in B^*$ such that $\phi(a) \neq \phi(b)$.)

2. State the Residue Theorem (from Complex Analysis) and use it to evaluate

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx \quad \text{for } a > 0.$$

Be sure to justify any limits required.

3. Consider a power series

$$\sum_{n=1}^{\infty} a_n x^n \tag{$\dagger$}$$

for real constants $a_n \in \mathbf{R}$. Show that there is a $\rho \in [0, \infty]$ such that either

(i) $\rho = 0$ by which we mean (†) converges only for $x = 0$, or

(ii) $\rho = \infty$ by which we mean (†) converges absolutely for all x, or

(iii) $0 < \phi < \infty$ and (†) converges absolutely if $|x| < \rho$ and diverges if $|x| > \rho$.

Give examples (with all $a_n \neq 0$) where $\rho = 0$, $\rho = \infty$ and $0 < \phi < \infty$.

4. Give a precise statement of the theorem which implies that a holomorphic function on an open subset of the complex plain is locally represented by a power series. Use your theorem to calculate the radius of convergence of the MacLaurin series for

$$f(z) = \frac{1}{1+e^z}.$$

(The MacLaurin series is just the Taylor series for f about $z = 0$.)

5. Let μ be a measure on the Borel sets of $\mathbf{R}$ such that for any Borel set $E \subseteq \mathbf{R}$ we have

$$\mu(E) = \inf\{\, \mu(U) : U \text{ is an open set containing } E \,\}$$

and $\mu([a, b]) < \infty$ for an interval $[a, b]$.

(i) Show that for any $\epsilon > 0$ there is an open set O and a closed set C such that $C \subseteq E \subseteq O$ and $\mu(O \setminus C) < \epsilon$.

(ii) Using the above, show that there are Borel sets G and F such that $F \subseteq E \subseteq G$ with $\mu(G \setminus F) = 0$.

(Hint: Finding a neighborhood O of E such that $\mu(O \setminus E) < \epsilon$ is pretty easy if $\mu(E) < \infty$.)

6. Show that a continuous function $f : (0, 1] \to \mathbf{R}$ is uniformly continuous if and only if there is continuous extension $g : [0, 1] \to \mathbf{R}$. (That is, g is a continuous function such that $g(x) = f(x)$ for all $x \in (0, 1]$.)