Written Certification Exam, Day 1

June 18, 2012, 9:00am – 12:00pm

(1) Let $p : X' \to X$ be a covering map, and assume that X' is path-connected. Let $x_0, x_1 \in X'$ and $x \in X$ be points such that $p(x_0) = x = p(x_1)$. Prove that the subgroups $p_*\pi_1(X', x_0)$ and $p_*\pi_1(X', x_1)$ are conjugate in $\pi_1(X, x)$.

(2) Ideals and quotients.
  (a) Find all ideals of the quotient ring $\mathbb{Q}[x]/(x^{14} - 1)$. In particular, how many such ideals are there?
  (b) Determine the structure of the quotient ring $\mathbb{Z}[x]/(5, x^2 - 2)$. Be as precise as you can.

(3) Of the following smooth manifolds, which ones admit a continuous nowhere vanishing vector field:
- S^2 minus a point.
- S^2
- S^3
- $S^1 \times S^1$
- $SL(n, \mathbb{R})$
- An oriented compact surface of genus three with no boundary.

(4) Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. Give three criteria (ϵ/δ, open sets, sequences) for f to be continuous on A. Show that these definitions are equivalent.

(5) Let Ω be an open connected subset of $\mathbb{C}$. Suppose that f_n is holomorphic[1] on Ω for each $n \geq 1$ and that the sequence $\{f_n\}$ converges to a function f uniformly on each compact subset of Ω.
  (a) Show that f is holomorphic on Ω.
  (b) Show that the sequence $\{f_n'\}$ of derivatives converges to f' uniformly on compact subsets of Ω.

(6) Let L be the splitting field over $\mathbb{Q}$ of $x^9 - 8$.
  (a) Determine the degree $[L : \mathbb{Q}]$ carefully explaining all conclusions.
  (b) Justify whether or not the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is abelian.
  (c) Justify whether or not the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is solvable.

[1]We say that g is holomorphic on Ω if $g'(z)$ exists for all $z \in \Omega$.

Written Qualification Exam, Day 2

June 19, 2012, 9:00am – 12:00pm

(1) Let the field K be an extension field of a field k. Show that there is a natural isomorphism of K-algebras $K \otimes_k \mathrm{M}_n(k) \to \mathrm{M}_n(K)$, where for a ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices over R.

(2) Denote by S^n the unit sphere in $\mathbb{R}^{n+1}$. If $F : S^n \to S^n$ is the antipodal map defined by $F(x) = -x$, then show by calculation, that the degree of F is $(-1)^{n+1}$.

(3) Let $C([0,1])$ be the complex vector space of continuous complex-valued functions on $[0,1]$.
   (a) Suppose that $\{f_n\}$ is a sequence in $C([0,1])$ and that f is a function on $[0,1]$ such that f_n converges uniformly to f. Show that $f \in C([0,1])$.
   (b) Assume without proof that
$$\|f\|_\infty := \sup\{\, |f(t)| : t \in [0,1]\,\}$$
   is a norm on $C([0,1])$. Show that $C([0,1])$ is a Banach space with respect to $\|\cdot\|_\infty$.

(4) Let T be a linear operator on a finite dimensional vector space V defined over a field k. Let $\chi_T(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_r)^{m_r}$ be the characteristic polynomial, and assume all the λ_i are distinct. Let V_i be the eigenspace corresponding to the eigenvalue λ_i.
   (a) Show that $\dim V_i \geq 1$ for all i, $1 \leq i \leq r$.
   (b) Choose nonzero $v_i \in V_i$. Show that $\{v_1, \ldots, v_r\}$ is linearly independent.
   (c) Conclude that if $\dim V_i = m_i$ for all i, then T is diagonalizable.

(5) Determine the singular homology groups of the standard torus (i.e., regarded as an identification space of a 2-dimensional rectangle) using the Mayer-Vietoris sequence.

(6) Let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a linear map.
   (a) Show that if T is bounded, then there is a linear map $S : \mathcal{H} \to \mathcal{H}$ such that $(Tv \mid w) = (v \mid Sw)$ for all $v, w \in \mathcal{H}$. (In other words, show that T has an adjoint.)
   (b) Conversely, show that if there is a (not necessarily bounded) map $S : \mathcal{H} \to \mathcal{H}$ such that $(Tv \mid w) = (v \mid Sw)$ for all $v, w \in \mathcal{H}$, then T is bounded.

Written Qualification Exam, Day 3

June 20, 2012, 9:00am – 12:00pm

(1) Show that any group of order 30 is the semidirect product of two smaller *abelian* groups.

(2) Let f be a complex function on an open connected subset Ω of the complex plane.
  (a) What are the Cauchy-Riemann equations for f at $z_0 \in \Omega$?
  (b) Discuss the existence of the complex derivative $f'(z_0)$ in terms of the Cauchy-Riemann equations at z_0. (Ideally, you should give both necessary as well as sufficient conditions for $f'(z_0)$ to exist. Note that you are not asked to prove anything here.)
  (c) Show that a real-valued function on Ω is holomorphic if and only if it is constant.

(3) Let $m > 1$ be a square-free integer, and $n \geq 1$ an *odd* integer. Let $F/\mathbb{Q}$ be any field extension with $[F : \mathbb{Q}] = 2$. Show that $x^n - m$ is irreducible in the polynomial ring $F[x]$.

(4) Let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a linear map.
  (a) What does it mean for T to be bounded?
  (b) Define the operator norm, $||T||$, of T and show that $||Th|| \leq ||T||\ ||h||$ for all $h \in \mathcal{H}$.
  (c) Show that T is bounded if and only if T is continuous from $\mathcal{H}$ to $\mathcal{H}$.

(5) Let ϕ_1 and ϕ_2 be two charts on $\mathbb{R}$ defined by $\phi_1(t) = t$ and $\phi_2(t) = t^3$. Are they C^∞ compatible? Prove your answer.

(6) Define the wedge product of two differential forms on a manifold. How does one use this operation to define the cup product of two de Rham cohomology classes? Prove that the cup product is well defined.