The tentative list of sample questions for the Topology Qual Exam

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Algebraic Topology

1. Let $f : S^n \to S^n$ be a continuous map, and let $\Sigma f : \Sigma S^n = S^{n+1} \to S^{n+1} = \Sigma S^n$ be its suspension. Show that f and Σf have the same Brouwer degree.

2. Show that there is no homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$ unless $n = m$.

3. Compute the homology groups of $\mathbb{C}P^n$ and show that the homology groups of $\mathbb{R}P^n$ are either 0 or cyclic.

4. Find the universal cover of $S^1 \vee S^1$, of $\mathbb{R}P^3 \vee S^1$, and of $S^2 \times S^2$.

5. Classify all threefold connected covering spaces of the figure-eight space $S^1 \vee S^1$.

6. **a** Show that a subgroup F' of a free group F is free.

   b If F has rank n and F' has finite index d, determine the rank of F'.

7. Prove that S^{n-1} is not a retract of B^n. Give an example of a deformation retract that is not a strong deformation retract.

8. Which of the following manifolds : $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, S^2, and the closed disk is the total space of a cover of a torus $S^1 \times S^1$

9. State and prove the path lifting property for covering spaces.

10. For finite CW complexes, define the Euler characteristic and prove the Euler-Poincare formula.

11. Calculate π_1 and H_1 of the Klein Bottle.

12. Prove that homotopy is an equivalence relation.

13. Given a short exact sequence of chain complexes

$$0 \to C' \to C \to C'' \to 0.$$

Define the connecting homomorphism $\Delta : H_n(C'') \to H_{n-1}(C')$ and prove exactness of the resulting long homology sequence.

14. If $f : (X, A) \to (Y, B)$ is a map of pairs of spaces such that $f(X) \subseteq B$, then show that $f_* = 0 : H_n(X, A) \to H_n(Y, B)$ in singular homology.

15. Define CW complex and show that $\mathbb{R}P^n$ is a CW complex.

16. State the excision axiom and use it to show that $\widetilde{H}_{i+1}(S^{n+1}) \approx \widetilde{H}_i(S^n)$.

17. Determine the homology of the torus using the Mayer-Vietoris sequence.

18. Let X be a path-connected space and $x_0, x_1 \in X$. Prove that $\pi_1(X, x_0) \approx \pi_1(X, x_1)$.

19. Outline the proof that all singular homology groups of a CW complex X with finitely many cells in each dimension are finitely generated.

20. Let X be a Hausdorff space and $x \in X$. Define the nth local homology group of X at x to be $H_n(X, X - \{x\})$. Prove that this is isomorphic to $H_n(U, U - \{x\})$ for any neighborhood U of x. Prove that any homeomorphism of the n-ball E^n maps the boundary sphere S^{n-1} onto itself.

21. If $f : S^n \to S^n$ is the antipodal map defined by $f(x) = -x$, then show that the degree of f is $(-1)^{n+1}$.

22. Show that for every integer k, there exists a map $f : S^n \to S^n$ of degree k.

23. Compute the Euler characteristic of the two-holed torus.

24. Define reduced homology $\widetilde{H}_n$ and show that $\widetilde{H}_n(X) = 0$ if X is a path-connected space.

25. State and prove the Five Lemma.

26. If A is a retraction, prove that the sequence (part of the long exact cohomology sequence of the pair (X, A)) is split, short exact

$$H^n(X, A) \to H^n(X) \to H^n(A).$$

27. Prove that $\pi_1(X \times Y) \approx \pi_1(X) \oplus \pi_1(Y)$.

28. State the Seifert-van Kampen Theorem and use it to show that $\pi_1(S^n) = 0$ for $n > 1$. Why doesnt the argument work for $n = 1$?

29. Assuming the existence of singular homology theory, define the chain complex which yields CW homology theory.

30. Which of the following spaces are contractible: (i) the solid torus $E^2 \times S^1$, (ii) $I \times I$, (iii) the subspace $E^n \times \{0\} \cup S^{n-1} \times I$ of $E^n \times I$, (iv) $\mathbb{R}^n - \{0\}$, (v) the open n-ball (interior of E^n), (vi) the set of all points $(x, y.z)$ in $\mathbb{R}^3$ such that $z^2 = x^2 + y^2$. Here $I = [0, 1]$.

31. Prove that a map $f : S^n \to X$ is homotopic to a constant map if and only if f can be extended to E^{n+1}.

32. Use the Meyer-Vietoris sequence to calculate the homology of $X \vee Y$.

33. Define a natural homomorphism $\alpha : H^n(X; G) \to$Hom$(H_n(X), G)$ and show that it is an epimorphism.

34. Let $p : X' \to X$ be a covering map and let $x_0, x_1 \in X'$ and $x \in X$ be points such that $p(x_0) = x = p(x_1)$. Prove that the subgroups $p_*\pi_1(X', x_0)$ and $p_*\pi_1(X', x_1)$ are conjugate in $\pi_1(X, x)$.

Differential Topology

1. Deduce the local structure of submersions (or the local structure of immersions, or the implicit function theorem) from the inverse function theorem.

2. Let M be an oriented smooth n-manifold, let $U \subset M$ be a coordinate neighborhood, and let ω be a smooth n-form whose support lies in U. Define the integral $\int_M \omega$ and show that your definition is independent of any choice of coordinates used.

3. Show that any smooth manifold admits a Riemannian metric.

4. Show that a smooth manifold has an orientable cover.

5. Let ω be a closed 2-form on $M = S^3 \times S^5$ and let $g : S^1 \times S^1 \to M$ be a smooth map. Prove that $\int_{S^1 \times S^1} g^*\omega = 0$.

6. Deduce the classical divergence theorem or the classical Stokes's theorem from the modern Stokes's theorem.

7. Let X be a vector field on a smooth n-manifold M, and suppose that $X_p \neq \mathbf{0}$ for some point $p \in M$. Show that there exist coordinates $x^1, \ldots, x^n$ in a neighborhood U of p such that $X|_U = \frac{\partial}{\partial x^1}$.

8. Which one of the following manifolds is a boundary of an oriented manifold: closed 3-ball, open 3-ball, Klein bottle, $S^1 \times S^1 \times S^1$.

9. Let $\mathbf{X} = \sin(2\pi x)\cos(2\pi y)\frac{\partial}{\partial x} + 4\frac{\partial}{\partial z}$ and $\mathbf{Y} = \sin(4\pi x)e^{\cos z}\frac{\partial}{\partial y} + 239\frac{\partial}{\partial z}$ be two vector fields on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Compute the Lie bracket $[\mathbf{X}, \mathbf{Y}]$.

10. Give an example of a non-integrable distribution on a manifold.

11. Give an example of a vector field whose flow is not defined for all time t.

12. Let X and Y be smooth vector fields on a smooth manifold M with flows given by Θ and Ψ respectively. Show that $[X,Y] \equiv 0$ if and only if for each $p \in M$ there is an $\epsilon_p > 0$ such that $\Theta_s \circ \Psi_t(p) = \Psi_t \circ \Theta_s(p)$ for all $|s|, |t| < \epsilon_p$.

13. Sketch a proof of the tubular neighborhood theorem for submanifolds of a compact manifold M.

14. Let M be a compact manifold, $\sigma : S^1 \to M$ be a nontrivial closed curve in M and $\ell > 0$. Explain how to put a Riemannian metric on M so that σ has length ℓ.

15. Please discuss the relationship between tangent vectors at a point and differentiation.

16. Define the tangent bundle of a manifold and explain its bundle structure.

17. Define the differential of a smooth map $F : M \to N$, where M and N are C^∞ manifolds. If $M = \mathbb{R}^n$, explain how this notion relates to the usual notion of derivative.

18. Prove: The set of 2×2 matrices of rank one is a submanifold of $M_{2,2}\mathbb{R}$

19. Give examples of conditions under which a vector field is complete.

20. Let M and N be C^∞ manifolds and $F : M \to N$ a C^∞ map.

    (a) What does it mean for vector fields X on M and Y on N to be F-related?

    (b) State and prove a relationship between the flows of F-related vector fields.

    (c) Show that a vector field X with flow ϕ_t is ϕ_t-invariant.

21. Discuss the notions of bracket $[X,Y]$ and Lie derivative $L_X Y$ of vector fields X and Y and state how the two notions relate. Prove: If $L_X Y = 0$, then the flows of X and Y commute.

22. Prove: Let V be a vector space and let $\alpha_1, \ldots, \alpha_k, \beta_1 \ldots, \beta_k \in V$. Prove that $\alpha_1 \wedge \cdots \wedge \alpha_k = \beta_1 \wedge \ldots \beta_k$ if and only if the α_i's and β_j's span the same subspace of V.

23. Define the notion of integration of a compactly supported n-form ω on a C^∞ oriented manifold M of dimension n, going through the following steps:

    - First assume ω is supported in a positively oriented coordinate neighborhood, and define its integral.
    - Explain why the definition in the previous step is well-defined; i.e., if one chooses another oriented coordinate chart containing the support, then the integrals agree.
    - Define the integral in the general case.

24. Suppose ω is an n-form on S^n that is the pull-back of a closed form on $\mathbb{R}^{n+1}$. What can you say about $\int_{S^n} \omega$?