Sample Algebra Questions

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1 Introduction

Graduate students often ask what they should study in preparation for a qualifying exam. The answer is of course that definitions, statements of theorems, examples and sketches of proofs of major theorems are the norm in a qualifying exam. However, this information is often not perceived by the student in the same way as it is intended by the faculty member.

To address this disparity, I have included below a number of sample questions to help you prepare for the algebra qualifying exam. These questions reflect only some of my own prejudices, and are not meant to reflect questions which other examiners might ask. These questions are in no sense intended to reflect a comprehensive review of the material, but should give you a good idea of the type and depth of question which you might be expected to answer. In particular, if you are not comfortable with the vast majority of the material reflected by these questions, you are not ready to take the qual.

In the exam itself, expect to be asked questions like those below, but also some which go beyond the bounds of your knowledge, and possibly outside the bounds of the syllabus. It is by probing the boundaries of your knowledge that we ascertain the depth of your knowledge. You are not expected to know the answer to everything we ask, but you are expected to know a majority.

2 Group Theory

1. What is meant by a group acting on a set? Give two characterizations and show that they are equivalent. Give two examples of a group action.

2. If a group G acts on a set S, define the notion of an isotropy subgroup and of an orbit. How are these notions related when the group and/or set are infinite? When both the group and set are finite, what more can be said?

3. Let G be a finite group and let H be a subgroup whose index in G is the smallest prime dividing the order of the group G. Show that H is a normal subgroup of G.

4. Derive the class equation. Use it to prove that the center of a (nontrivial) p-group is nontrivial.

5. Show that if Z is the center of a group G, and G/Z is cyclic, then G is abelian.

6. Show that if p is a prime, any group of order p^2 is abelian.

7. Let G be a group and G' its commutator subgroup, that is G' is the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$. Show that

   (a) $G' \triangleleft G$

   (b) G/G' is abelian

   (c) If $H \triangleleft G$ and G/H is abelian, then $G' \subset H$.

8. Show that if G is a group and $N \triangleleft G$, then G is solvable if and only if N and G/N are solvable.

9. Show that the following two statements are equivalent:

   (a) Every finite group of odd order is solvable.

   (b) Every finite nonabelian simple group has even order.

10. Suppose that p is a prime, and that G is a nonabelian group of order p^3. Let Z denote the center of G. Show that

    (a) $G' = Z$, and

    (b) G/Z is the direct product of two cyclic groups of order p.

11. Show that a finite abelian group has a subgroup of every order dividing the order of the group.

12. Show that a finite group always has a composition series.

13. Suppose that an arbitrary group G has two subgroups H and K each of finite index in G. Show that the intersection $H \cap K$ has finite index in G; in particular, show that $[G : H \cap K] \leq [G : H][G : K]$.

14. Let $p < q$ be primes, and let G be a group of order pq. Show that if $p \nmid (q-1)$ then G is cyclic.

15. Let G be a finite group, and p a prime dividing the order of G. Let $H = \{x \in G | o(x) = p^m, \text{ some } m\}$. Show that H is the union of all the p-Sylow subgroups of G.

16. Show that a finite cyclic group has a unique subgroup of every order dividing the order of the group.

17. State the structure theorem for finitely generated abelian groups. Characterize up to isomorphism all abelian groups of order 72 both in terms of elementary divisors and invariant factors.

18. Show that every finite abelian group is the direct product of its Sylow p-subgroups.

19. Show that every group of order 56 is the semidirect product of its Sylow p-subgroups.

20. Let G be a finite group and H a normal subgroup. If $\gcd(|H|, [G : H]) = 1$, show that H is the unique subgroup of G with order $|H|$.

21. State the Jordan-Hölder theorem. Give an example of two nonisomorphic groups with the same composition factors.

22. Why is the classification of all the finite simple groups such a big deal? That is, how does it fit into the grand scheme of classifying all the finite groups? Hint: Suppose that G is a finite group with composition series $G \triangleright K \triangleright \{1\}$. This gives us a natural short exact sequence: $1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1$ in which we have information about K and G/K. Now go read about the "extension problem", and think about the general case.

3 Ring Theory

1. Define the terms integral domain, irreducible element, and prime element.

   (a) Are there rings with no irreducible elements?

   (b) Are there commutative rings in which irreducible elements are not prime?

   (c) Prove that in an integral domain, prime elements are irreducible.

2. Give examples of a noncommutative ring with zero divisors, a noncommutative division ring, and integral domain, a UFD, a PID, a Euclidean domain and examples which show that ID $\not\Rightarrow$ UFD $\not\Rightarrow$ PID $\not\Rightarrow$ ED. Be sure to justify that your examples have or do not have the requisite properties.

3. Define what is meant by a Noetherian ring. Give three equivalent conditions and demonstrate their equivalence.

4. Why isn't every integral domain a UFD? That is, what goes wrong when one tries to factor nonzero nonunits. What conditions does one need to impose? Hint: there are separate conditions which guarantee the existence of a factorization and uniqueness of a factorization.

5. What is the characteristic of a commutative ring with identity? What can one say about the characteristic of an integral domain?

6. Prove that a Euclidean integral domain is a PID.

7. Show that a PID is a UFD.

8. Let A be a UFD, and X an indeterminate. Show that any irreducible in A remains irreducible in $A[X]$.

9. Show that $k[x_1, x_2, \ldots]$ is a non-Noetherian UFD. Be careful here. The Noetherian issue is trivial, but the UFD part is not. For example what are the irreducibles? Does a polynomial which is irreducible in $k[x_1, \ldots, x_n]$ remain irreducible in $k[x_1, x_2, \ldots]$?

10. Prove that over a field K, a polynomial of degree n has at most n roots in any splitting field. Does this remain true if the field K is replaced by a division ring, like Hamilton's quaternions? Why or why not?

11. Let A be a UFD with quotient field K, and let $f \in A[X]$. Show that f is irreducible over A iff f is primitive and irreducible over $K[X]$.

12. Let A be a UFD with quotient field K, and let L be a field. Let $f \in A[X]$, $\deg(f) = r \geq 1$, and let $\sigma : A \to L$ be a ring homomorphism. Show that if $\deg(f^\sigma) = \deg(f)$ and f^σ is irreducible in $L[X]$, then f is irreducible in $K[X]$. In particular, if f is primitive, then f is irreducible in $A[X]$.

13. Use the previous question to show that $f(x) = x^3 + 3x^2 + 5x + 2$ is irreducible over $\mathbb{Z}$. (Let $\sigma : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ be the natural map)

14. State and prove Eisenstein's criterion for irreducibility.

15. State and prove the Chinese Remainder theorem in the context of a commutative ring with identity. Give an interpretation in terms of $\mathbb{Z}/N\mathbb{Z}$ and of $(\mathbb{Z}/N\mathbb{Z})^\times$. (Recall that if I, J are ideals in a commutative ring that $IJ \subset I \cap J$ and $(I \cap J)(I + J) \subset IJ$. Now use comaximality)

16. Let A be a ring, and X an indeterminate. For each property P listed below, consider the question: "If A has P, does $A[X]$?" If so, give a proof; if not a counterexample.

    (a) integral domain

    (b) PID

    (c) UFD

    (d) Noetherian ring

17. Define the notion of prime and maximal ideals. Give characterizations of each concept in terms of quotient rings. Give an example to show that not all prime ideals are maximal.

    (a) Is (x) prime/maximal is $\mathbb{Z}[x]$, $\mathbb{Q}[x]$?

    (b) If k is a field, is (x) prime or maximal in $k[x, y]$?

    (c) Characterize the maximal ideals of $\mathbb{Q}[x]$, or $\mathbb{C}[x]$.

    (d) Is $(x - 3)$ prime/maximal in $k[x, y]$?

18. Define what is meant by the localization of a ring with respect to a multiplicative set. What does localizing at a prime mean? Describe the localization of the ring $\mathbb{Z}[x]$ at the prime ideal (x). Is (x) maximal in $\mathbb{Z}[x]_{(x)}$? If so, describe the field to which its quotient is isomorphic. How does $\mathbb{Z}[x]_{(x)}$ compare to $\mathbb{Q}[x]_{(x)}$?

19. Show that every ideal of the localized ring $S^{-1}A$ is of the form $S^{-1}I$ where I is an ideal of A. Use this to show that if A is a PID, then so is $S^{-1}A$. (Hint: if $[a, s]$ is in an ideal of $S^{-1}A$, then so is $[a, 1]$ and hence so is $[a, t]$ for every $t \in S$.)

20. If A is a commutative ring with identity and the polynomial ring $A[x]$ is Noetherian, does it follow that A is Noetherian? This is the converse of a well-known theorem.

4 Modules and Linear Algebra

1. Discuss the notion of a minimal polynomial of a linear transformation. Does it have to be irreducible?

2. Let V be a finite-dimensional vector space over the field k, and let $T \in \mathrm{End}_k(V)$. Show how to use T to make V into a finitely generated torsion $k[x]$-module.

3. State the basic decomposition theorem for finitely generated modules over PIDs and apply it to the situation in the previous question.

4. Explain the origins of the rational canonical form of a matrix.

5. Describe what is meant by the Jordan form of a matrix.

6. Find all rational and Jordan canonical forms of a matrix in $M_5(\mathbb{C})$ having minimal polynomial $x^2(x-1)$. Be sure to give the corresponding invariants and the characteristic polynomial.

7. Show that any linear operator T on a finite dimensional vector space (over a field of characteristic not equal to 2) which satisfies $T^2 = I$ is diagonalizable.

8. State and prove the rank-nullity theorem for finite dimensional vector spaces.

9. For a vector space over a field, we know that any linearly independent set can be extended to a basis, and that any spanning set can be reduced to a basis. Let M be a free module of finite rank over a PID R. Are the corresponding statements for M valid? Proof or counter example.

10. Let M be a free module of finite rank over a PID R and N a submodule. What does the elementary divisor theorem say about bases of N versus M?

11. Let M and N be modules over a commutative ring R.

    (a) Characterize the tensor product $M \otimes_R N$ as a universal object in some category.

    (b) Give a construction for the tensor product $M \otimes_R N$.

    (c) Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ if $\gcd(m, n) = 1$.

    (d) Characterize $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ in general.

    (e) If G is a finite abelian group, and $G \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$ for all primes p, show that $G = 0$. Does the result remain true if G is infinite.

12. Let A be a commutative ring with identity, and suppose that $0 \to M \to N \to F \to 0$ is a short exact sequence of A-modules. Show that if F is free, the sequence splits. Find examples of A, M, N, F so the short exact sequence doesn't split. What are the minimal conditions to place of F so that the sequence splits? What can be said about the relationship between M, N and F.

13. Give an example of a free-module and a submodule which is not free. What conditions guarantee that a submodule of a free module be free?

5 Field Theory

1. Let E/F be an extension of fields, and let $\alpha \in E$. Define what is meant by the phrase α is algebraic over F. What is meant by saying that E/F is an algebraic extension?

2. What is meant by the minimal polynomial of an element algebraic over F? Why is $F[\alpha] \cong F(\alpha)$ when α is algebraic over F?

3. If α is algebraic over F, what is the degree of $F(\alpha)/F$? Exhibit a basis and prove that it is a basis.

4. Show that an extension of fields E/F is finite if and only if it is algebraic and finitely generated.

5. Consider a tower of fields $K \subset F \subset E$. Show that E/K is finite (resp. algebraic) if and only if E/F and F/K are finite (resp. algebraic). Consider the situation for normal extensions.

6. Give three equivalent conditions for an extension to be normal (don't restrict to finite extensions), and demonstrate their equivalence.

   (a) Every embedding $\sigma : E/F \to \bar{F}$ is an automorphism of E.

   (b) E is the splitting field of a family of polynomials in $F[X]$.

   (c) Every irreducible polynomial in $F[X]$ with one root in E splits in E.

7. Show that finite fields are perfect.

8. Let F be a field, and $f \in F[x]$ be irreducible. Show that all the roots of f occur with the same multiplicity. If F is perfect, show that all roots occur with multiplicity one.

9. Prove that the Galois group of a finite extension of finite fields is cyclic.

10. Compute the Galois group of the splitting field of $x^8 - 1$ over $\mathbb{Z}_3$. Is $x^8 - 1$ separable over $\mathbb{Z}_3$? Do you really need to check? Hint: $x^9 - x = x(x^8 - 1)$.

11. Describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ and discuss the Galois correspondence. Here ζ_n is a primitive nth root of unity.

12. Describe all the fields which lie between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{12})$.

13. Define what is meant by a purely inseparable extension of fields. If $k(\alpha)/k$ is purely inseparable, verify that any embedding $\sigma : k(\alpha) \to \bar{k}$ (over k) is the identity map.

14. Let $f \in F[x]$ be a nonconstant polynomial. Show that there is a field extension of F in which f has a root.

15. Describe the construction of an algebraic closure of a field.

16. Show that any finite subgroup of the multiplicative group of a field is cyclic.

17. Let E/F be an extension of fields and let $\alpha \in E$ be algebraic over F. If $[F(\alpha) : F]$ is odd, show that $F(\alpha) = F(\alpha^2)$.

18. Suppose that ξ is transcendental over a field F. Let E be a subfield of $F(\xi)$ which is not equal to F.

    (a) Show that ξ is algebraic over E.

    (b) Show that $F(\xi)$ is a finite extension of E.

    (c) Is E algebraic over F?

19. Compute the Galois group of $x^6 + 27$ over $\mathbb{Q}$.

20. Compute the Galois group of $x^8 - 16$ over $\mathbb{Q}$.