Topology

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General Topology

Some basic ideas from general topology, while not tested explicitly, are regarded as prerequisites for qualification in topology and will be assumed during the qualifying exam. These ideas include: the definitions and basic facts about topological spaces, bases, neighborhoods, continuous functions and homeomorphisms; connectedness, local connectedness and path-connectedness; compactness, local compactness and compactness in Euclidean space; Hausdorff spaces, normal spaces and metric spaces; the quotient topology and the product topology.

Algebraic Topology

a. Elementary Homotopy and the Fundamental Group: Basic homotopy of maps, deformation retracts, homotopy equivalences and homotopy type. The fundamental group and its main properties. Computation of the fundamental group. The theory of covering spaces and its relation to the fundamental group.

b. Homology Theory: Construction and basic properties of singular homology theory including excision, the homotopy property and exactness. The Eilenberg-Steenrod axioms. CW complexes and cellular homology theory. Computation of homology groups. Applications of homology theory. Elementary cohomology theory.

c. Homological Algebra: Exact sequences, chain and cochain complexes, chain homotopy and the exact homology sequence of a short exact sequence of chain complexes. Introduction to categories and functors.

Theorems: There are two types of theorems: The first consists of theorems whose proofs are intricate and the student is expected to be able to state them and apply them, but is not expected to prove them. The second type consists of theorems which the student is expected to know how to prove. In the first type are: Van Kampen's Theorem, the existence of a universal cover, the excision property of singular homology theory, the isomorphism of singular and cellular homology. In the second type are: Lifting theorems for covering spaces, calculation of the fundamental group of the circle, the Mayer-Vietoris sequence, the Euler-Poincare formula, the Brouwer fixed point theorem.

Parts **b** and **c** are normally covered in detail in Math 114. Part **a** is often covered in Math 74; it can also be learned by reading independently under the direction of the student's committee.

Differential Topology

Smooth manifolds. Submanifolds, product manifolds, boundary of a manifold and examples of these. The tangent space, cotangent space and the differential of a smooth map. Embeddings, immersions, submersions and diffeomorphisms. Applications of partitions of unity. Vector bundles, vector fields and flows. Orientability. The Lie derivative and the Lie bracket. Differential forms and operations on them. Integration on manifolds. Applications of the Stokes's Theorem to cohomology.

Theorems: Theorems which the student is expected to state and apply: the inverse function theorem, the existence of partitions of unity, the existence and uniqueness of flows of vector fields and their properties, the general theorem of Stokes. Theorems which the student is expected to be able to prove: the theorem on rank, the existence of a Riemann metric on a manifold, the theorem on embedding of a closed manifold into $\mathbb{R}^n$.

The material on differential topology is generally covered in Math 124, which assumes as undergraduate preparation a course on analysis on manifolds at the level of Spivak's "Calculus on manifolds". Students who do not have this background should normally enroll during the first year in Math 73, which furnishes the necessary prerequisites for Math 124.

References

The student is not expected to read all the books on the list. The major references are indicated with an asterisk. If in doubt, the student should consult with a faculty member to determine which sources cover which material.

General Topology

1. Dugundji, *Topology*
2. *Munkres, *Topology*
3. Willard, *General Topology*

Algebraic Topology

1. Fulton, *Algebraic Topology: A First Course*
2. *Hatcher, *Algebraic Topology*
3. *Massey, *A Basic Course in Algebraic Topology*
4. Spanier, *Algebraic Topology*

Differential Topology

1. *Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*
2. Guillemin and Pollak, *Differential Topology*
3. *Lee *Introduction to smooth manifolds.*
4. *Spivak, *Calculus on Manifolds* (This should be read first.)
5. Tu, *An Introduction to Manifolds*