4/2007

DARTMOUTH COLLEGE
DEPARTMENT OF MATHEMATICS
GRADUATE PROGRAM

COMBINATORICS: *Syllabus for Graduate Certification*

Because of the unusual breadth of combinatorial mathematics, the topics in the subject relevant to the graduate certification requirement have been divided into primary and secondary topics. The student is required to have a basic knowledge of each of the three primary topics and at least one of the secondary topics. The primary topics are (1) Classical Combinatorics, (2) Algebraic Techniques, and (3) Graph Theory. The secondary topics include (A) Ordered Sets, (B) Coding Theory, (C) Combinatorial Geometry and Matroids, (D) Matching Theory, (E) Random Graphs and the Probabilistic Method, (F) Symmetric Functions, and (G) Representations of the Symmetric Group. Other topics could serve as secondary topics with the approval of the GEC in consultation with the combinatorics faculty.

PRIMARY TOPICS

1. **Classical Combinatorics**
   Counting functions: arbitrary, injective or surjective functions with domain and range either distinguishable or indistinguishable.
   Classical Enumeration: sets, multisets, permutations, multiset permutations, partitions, set partitions, and compositions; Applications to Bell numbers, Stirling numbers of the first and second kinds, and Eulerian numbers; Producing and solving recurrence relations, bijective methods in proofs, with applications of these techniques to the Catalan and Fibonacci families.
   Finite Configurations: Latin squares; Balanced and incomplete block designs, symmetry, and construction techniques; Relationships of the design parameters; Finite, affine, and projective geometries and their relations to designs and Latin squares.

2. **Algebraic Techniques**
   Generating functions: The algebra of the formal power series ring $\mathbb{C}[[x]]$; Ordinary and exponential generating functions; Applications to partition problems, Stirling numbers of the first kind, permutation statistics, and Bell numbers.
   Group actions: Counting orbits using the Burnside Lemma; Polya's Theorem with at least one significant application such as counting unlabeled graphs or trees.
   Sieve techniques: The Fundamental Theorem of Möbius inversion and its relation to number theoretic Möbius inversion and to the principle of inclusion and exclusion; Computation of the Möbius function of simple partially ordered sets, of the set partition lattice, and of the subspace lattice; At least one significant application such as counting labeled connected graphs.

3. **Graph Theory**
   Basic concepts: Euler and Hamiltonian cycles; Connectedness, n-connectedness, and Menger's Theorem (both point and line forms); Adjacency matrix and incidence matrix; Bipartite graphs; Tournaments.
   Topology: Isomorphism; Planarity and the planar dual; Line graphs; Euler's formula and related inequalities for planarity; Graph coloring including the Four and Five Color Theorems; Deletion/Contraction and the chromatic polynomial.
   Networks: Flows, cuts, and flow-augmenting paths; The Max-Flow Min-Cut Theorem; Integral flows and Menger's Theorem.

SECONDARY TOPICS

A. **Ordered Sets**

   Partial orders and lattices; Distributive, complemented, geometric, and modular lattices; The Fundamental Theorem on Finite Distributive Lattices; Birkhoff's Theorem for uniquely complemented lattices; Chains and antichains; Sperner's Theorem; The incidence algebra; Basic concepts of dimension theory; Interval orders.

B. **Coding Theory**

   Linear and nonlinear block codes; Bounds on the size of codes of length n and minimum distance d; The major properties of Hamming, BCH, Reed-Muller, Golay, Cyclic, and Quadratic Residue codes; Dual codes, the MacWilliams Theorem on the relationship between the weight enumerator of a code and its dual code, and applications of this theorem; Relationship between codes and designs.

C. **Combinatorial Geometry and Matroids**

   Matroids and combinatorial geometries; Equivalence of definitions by closures, circuits, bases, the greedy algorithm, and rank functions; Connectivity; Geometric lattices; Modular geometric lattices and projective geometries; The lattice of flats of a combinatorial geometry; Relationship of combinatorial geometries and matroids to graphs; Representable and nonrepresentable geometries.

D. **Matching Theory**

   Hall's Theorem on systems of distinct representatives and its generalizations; Dilworth's Theorem with application to Birkhoff's Theorem on doubly stochastic matrices; König's Theorem on 0-1 matrices; Relations between SDR's, 0-1 matrices, and Latin squares; The assignment problem and algorithms for finding SDR's.

E. **Random Graphs and the Probabilistic Method**

   Probabilistic models for graphs; Markov's inequality, Chebyshev's inequality, and the second moment method; Threshold functions with at least one application; The evolution of random graphs; The probabilistic method with application to diagonal Ramsey numbers; The Lovász Local Lemma with application to diagonal Ramsey numbers.

F. **Symmetric Functions**

The graded ring of symmetric functions; Descriptions of the five natural bases: monomial, elementary, complete homogeneous, power sum, and Schur symmetric functions; Relationships between the five bases; Jacobi—Trudi determinants; The Hall involution and its properties; The Hall inner product and orthogonality; The Robinson—Schensted—Knuth correspondence and its properties.

G. **Representations of the Symmetric Group**

Matrix representations; G-modules and the group algebra of a finite group; Reducibility; Maschke's Theorem; Schur's Lemma; Irreducible characters and orthogonality; Decomposition of the group algebra; Restriction and induction; Young subgroups; Tableaux and tabloids; Kostka numbers; Young's rule.

REFERENCES

1. Classical Combinatorics
Bogart, *Introductory Combinatorics, Second Edition* (Chapters 1,2,6)
Graham, Rothschild, and Spencer, *Ramsey Theory* (Chapter 1)
Liu, *Introduction to Combinatorial Mathematics*
Riordan, *Combinatorial Mathematics* (Chapters 2—4)
Stanley, *Enumerative Combinatorics, Volume 1* (Chapter 1)

2. Algebraic Techniques
Aigner, *Combinatorial Theory*
Bogart, *Introductory Combinatorics, Second Edition* (Chapters 3, 8)
Flajolet and Sedgewick, *Analytic Combinatorics* (Chapters I-II)
Stanley, *Enumerative Combinatorics, Volume 1* (Chapters 1—3)
Wilf, *Generatingfunctionology*

3. Graph Theory
Bogart, *Introductory Combinatorics, Second Edition* (Chapters 4,5)
Bollobas, *Graph Theory: An Introductory Course*
Bondy and Murty, *Graph Theory With Applications*
Golumbic, *Algorithmic Graph Theory and Perfect Graphs*

A. Ordered Sets
Birkhoff, *Lattice Theory*
Davey and Priestley, *Introduction to Lattices and order*
Bogart, *Introductory Combinatorics, Second Edition* (Chapter 7)
Stanley, *Enumerative Combinatorics, Volume 1*
Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory* (Chapter 3)

B. Coding Theory
Berlekamp, *Algebraic Coding Theory*
Peterson and Weldon, *Error-Correcting codes*

Pless, *Intorduction to the Theory of Error-Correcting Codes*
Sloane and MacWilliams, *The Theory of Error Correcting Codes*
Van Lint, *Coding Theory*

C. Combinatorial Geometry and Matroids
Aigner, *Combinatorial Theory*
Crapo and Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*
Oxley, *Matroid Theory*
Welsh, *Matroid Theory*

D. Matching Theory
Bogart, *Introductory Combinatorics, Second Edition* (Chapter 5)
Hall, *Combinatorial Theory*
Mirsky, *Transversal Theory: An Account of Some Aspects of Combinatorial Mathematics*
Ryser, *Combinatorial Mathematics*

E. Random Graphs and the Probabilistic Method
Alon and Spencer, *The Probabilistic Method*
Bollobas, *Random Graphs*
Palmer, *Graphical Evolution*
Spencer, *Ten Lectures on the Probabilistic Method*

F. Symmetric Functions
Macdonald, *Symmetric Functions and Hall Polynomials* (Chapter 1)
Sagan, *The Symmetric Group* (Chapters 3,4)

G. Representations of the Symmetric Group
Sagan, *The Symmetric Group* (Chapters 1,2)
Stanley, *Enumerative Combinatorics, Volume 2* (Chapter 7).