Syllabus for Applied Mathematics Graduate Student Qualifying Exams, Dartmouth Mathematics Department

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August 12, 2011

We aim to touch upon many topics that a professional applied mathematician should know; thus an applied qual is broader and less deep than most pure mathematics quals. We split the qual topics below in to two broad "flavors": i) numerical analysis with PDE, and ii) complex systems with data analysis. One could be chosen in its entirely, or the other, or an appropriate mixture of both, or a mix with other special topics. It is possible to do *two* applied quals, one of each flavor, or in discussion with the committee, on special topics. There is a lot of flexibility in the choice of content, steered by the interests of the student and the faculty committee. The final level and amount of content tested will also be adjusted based on the *prior background of the student* (in order to create a reasonable qual preparation workload). In *italics* we include optional topics that a keen student might add.

Format: written, oral, or computer coding, or any combination, as agreed by the committee and student.

Prerequisites: We will assume knowledge of the undergraduate material Math 23 (differential equations [1, up to Ch. 10]); Math 22/24 (linear algebra); Math 20 (basic probability); Math 63 (real analysis); Math 43 (complex analysis). We assume some basic functional analysis such as Banach and L^p spaces, weak convergence, compact operator. Several other qual topics below we expect may have been touched upon in a typical student's undergraduate courses.

We try to list only high-quality references in the below, i.e. book sections known to be good, self-contained introductions.

TO DO: We still need refs for data analysis, Markov chains, or cut MC altogether.

1 Material common to both flavors

1.1 Applied and numerical linear algebra

- Matrix norms [14, Ch. 1–3].

- Review four fundamental spaces of a general matrix [13, 3.6]; also at `http://math.mit.edu/linearalgebra/ila0306.pdf`
- Singular value decomposition (SVD): definition, uniqueness (not full proof, but sketch thereof), properties: best rank-k approximation to a matrix [14, Ch. 4–5].
- QR decomposition, and least-squares solution of a linear system via SVD or QR [14, Ch. 6–11].
- Linear systems: basics of Gaussian elimination. [14, Ch. 20–21]
- Principal Component Analysis, implementation via SVD [10]

Example questions:

1. Prove that the spectral radius of a matrix is no larger than its 2-norm.
2. Suppose a_{ij} represents the response of subject j to drug i. Perform PCA using a software package and interpret the results.
3. Sitting at a Matlab terminal, extract a basis for the column space of a given matrix.
4. If on a certain computer it takes 1 sec to solve a square linear system of size 10^3, how long would it take to solve a system of size 10^4? Discuss your assumptions.

1.2 Computer programming

This is hard to assess as formally as the other topics, but an applied Ph.D candidate should be able to develop computer programs that use the following concepts competently, in at least one language (e.g. C, fortran, Matlab, Mathematica). Math 26 covers such skills, as well as some of the numerical computing below.

- Loops, vectors and matrices, conditionals, input/output from data files, subroutines
- Documentation of codes and subroutine purposes and interfaces
- Use of library routines
- Communicating results and data via graphs, or images, with appropriately chosen axes (e.g. linear/logarithmic)

2 Flavor I: Numerical analysis and PDEs

2.1 Linear differential equations

Some of the below is covered by the 2nd half of Math 46 [11, Ch.6].

- Sturm-Liouville eigenvalue problems [1, Ch.11], [12, Ch.1], energy method proof for definite sign of eigenvalues [11, Ch.4]
- Classifying 2nd-order PDEs. Review separation of variables.
- Laplace equation and boundary-value problem (BVP): maximum principle, energy method for proving uniqueness [11, Ch.6] [3, Ch.2], fundamental solution in $\mathbb{R}^n$, Poisson kernel for Laplace equation in ball in $\mathbb{R}^n$, Laplace's equation in a half-space given Cauchy data. See [11, Ch.6], then [2, core parts of Ch.1–4] and [3, Ch.1–2]. Add a sprinkling of [4].
- Heat equation in $\mathbb{R}^n$, general solution via fundamental solution.
- Wave equation: d'Alembert's solution in 1D, fundamental solution in $\mathbb{R}^3$, Hadamard's method of descent for fundamental solution in $\mathbb{R}^2$ (see unpublished notes on Barnett webpage).
- Concept of Green's function in 1D [11, Ch. 4] [12, Ch.1]: application to nonhomogenous solution of PDEs. Conservation laws [11, Ch.6] and application to derivation of cylindrically- and spherically-symmetric Laplacian operator.

Optional topics: Basics of Sobolev spaces and elliptic PDE, existence via Lax-Milgram in H^1 [3, Ch.5–6]; distributions; Cauchy problem for general coefficient hyperbolic PDE in 2 variables [5]; Green's Representation Theorem [8, Ch.6]; potential theory and application to existence of solutions to elliptic PDE.

Example questions:

1. Prove uniqueness for the interior Dirichlet BVP for Laplace's equation in two ways (maximum principle, and energy method). What changes for the exterior BVP?
2. Give an integral form of the solution to the initial boundary value problem for the heat equation on $\mathbb{R}^n$
3. Give examples of well-posed and ill-posed PDEs. *ans: e.g. forward and backward heat equation on* $\mathbb{R}$

2.2 Fourier series, transforms, and integral equations

Most of this topic is covered at a good advanced undergrad or first-year grad level by [11, Ch.4] and [12, Ch.3]. It is also a large chunk of the Math 46 syllabus.

- Fourier series. Their convergence theorem (via the Riemann-Lebesgue lemma) [2].
- Integral equation classification: Volterra, Fredholm, basic solution methods in one variable; expression of ODEs and Sturm-Liouville problems as IEs [11, Ch. 4]
- Neumann series, Picard's method for nonexistence of eigenvalue of Volterra; application to Picard-Lindelöf theorem for existence and uniqueness of ODE solution in some interval [7, Sec. 10.1]
- Eigenfunction expansions: Hilbert-Schmidt theorem, Green's function, integral operator as inverse of differential operator [11, Ch.4].
- Fourier transform in $\mathbb{R}^n$ at level of [11, Ch. 6] or [4, preliminaries], application to solution of linear constant-coefficient PDEs such as a BVP in $\mathbb{R} \times (0, \infty)$ [11, Ch.6.5.2]

Optional topics: Fredholm alternative theorem for compact operators (see: [7, Ch.3], [3, App.D]).

Example questions:

1. Solve $u_t = u_{xx}$ on $x \in \mathbb{R}$, $t > 0$, $u(x, 0) = e^{-x^2}$ analytically. *ans: Use Fourier transform method*
2. Prove that Volterra equations have no eigenvalues.

2.3 Numerical analysis

Math 26 / Engineering 91 would be a good starting point for the below. Some is also covered in Barnett's incarnations of Math 116/126.

- Conditioning, floating point approximation, backwards stability of algorithms [14, Ch.12–14]
- Simple iterative methods for nonlinear equations: Newton's method [7, Ch. 6.2].
- Polynomial interpolation, quadrature, up to Gaussian quadrature with theorems, and spectral convergence of periodic quadrature [7, Ch. 8–9].
- Time-stepping for ODEs: explicit, implicit, stability, definition of consistent scheme. Order of convergence for simple schemes, forward Euler, etc. Lax theorem: "consistency + stability = convergence". [7, 10.2–4]
- Simple 1D finite difference methods for a 2-point BVP, convergence rate [7, Ch.11]

Optional topics: optimization via Quasi-Newton methods in $\mathbb{R}^n$, fast Fourier transform, other chapters of [14], [7].

Example questions:

1. Set up the simplest finite-difference approximation to solve the BVP $u'' + u = 1$ on $[0, 1]$ with Dirichlet boundary conditions. Discuss convergence rate.

2. Explain how you would approximate $\int_0^{2\pi} e^{a \cos\theta} d\theta$. What convergence rate would you expect? Repeat if the upper limit is instead 1.

3. An exact solution is $\mathbf{y} = F(\mathbf{x})$ given data $\mathbf{x}$, and $\tilde{F}$ is an approximate implementation of F. Explain what it means for $\tilde{F}$ to be stable, or backwards stable. Does this imply $\tilde{F}(\mathbf{x})$ is close to the exact answer? When?

4. Ex. 15.1 of [14]

3 Flavor II: Complex systems and data analysis

3.1 Statistics and data analysis

- Undergradute probability and statistics, i.e. first half of Math 50 syllabus [9]. Variance, covariance matrix.
- Bayesian inference (at level of Math 50 [9]).
- Multivariate Gaussian pdfs and their marginal and conditional pdfs [6, App.E]
- Least-squares fitting of data to a model with parameters; maximum likelihood
- Clustering: k-means
- Clustering: Spectral clustering
- Multidimensional scaling

Optional topics: Spectral analysis, discrete Fourier transform, FFT, power spectrum, wavelets, time-frequency transform (spectrogram).

Example questions:

1.

3.2 Network analysis and discrete applied math

- The Graph Laplacian: definition, spectral properties, diffusion on a graph.
- Partition Decompling Method
- Markov chains: eigenvectors, graph, theorems for uniqueness of steady-state

Optional: Metropolis algorithm for estimating high-dimensional integrals, convergence rate.

Example questions:

1.

References

[1] W. E. Boyce and R. C. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. Wiley, 8th ed. edition, 2004.

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[3] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

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[6] E. Jaynes. *Probability theory: the logic of science*. Cambridge University Press, 2003.

[7] R. Kress. *Numerical Analysis*. Graduate Texts in Mathematics #181. Springer-Verlag, 1998.

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[9] R. J. Larsen and M. L. Marx. *An Introduction to Mathematical Statistics and its Applications*. Prentice-Hall, 4th edition, 2002.

[10] G. Leibon, S. Pauls, D. N. Rockmore, R. Savell, and M. Herron. *Statisical Learning for Complex Systems: an Example-driven Introduction*. 2010.

[11] D. Logan. *Applied Mathematics*. Wiley, 4th edition, 2006.

[12] I. Stakgold. *Boundary Value problems of mathematics physics. Volumes I & II*. SIAM Classics in Applied Math #29, 1967.

[13] G. Strang. *Introduction to Linear Algebra*. 4th ed. edition, 2009.

[14] L. N. Trefethen and D. Bau III. *Numerical Linear Algebra*. SIAM, 1997.