ALGEBRA ORAL QUALIFYING EXAM SYLLABUS

Abstract algebra is the study of fundamental algebraic structures that occur throughout mathematics: groups, rings, modules, categories, and fields—as well as maps between them. A solid grasp of algebra can be useful to students of all mathematical stripes, and accordingly the suggested syllabus is a "strong core plus more" model: a student preparing for an oral qualifying exam in algebra should (1) feel confident with the core topics (covered in most offerings of 101/111) and (2) should select a supplementary advanced topic for examination, in consultation with the chosen oral exam committee.

The core and some suggested supplementary topics are listed below.

1. Core topics covered in most offerings of 101/111

(1) Groups (e.g., [4, Chapters 1–5], [15, Chapters 1-6])
  - (a) Cyclic, abelian, non-abelian groups; normal subgroups and quotient groups; normal series, solvable groups, commutators, composition series, Jordan-Hölder theorem; the extension problem
  - (b) Group actions, G-sets and structure theorem, Sylow theory, semi-direct products and split exact sequences
  - (c) Symmetric group, alternating subgroup, cycle decomposition and conjugacy classes

(2) Rings (e.g., [4, Chapter 7–9], [10, Chapters 2,4], [15, Chapters 7-8])
  - (a) Rings: examples, properties, homomorphisms, group rings, polynomial rings, division algorithm, polynomials versus polynomial maps
  - (b) Ideals: prime and maximal ideals, operations on ideals, Chinese remainder theorem (CRT)
  - (c) Irreducibles and prime elements, UFDs, PIDs, Noetherian rings, Euclidean domains, Gauss's lemma and corollaries, irreducibility tests, Hilbert's Basis Theorem, cyclotomic polynomials

(3) Modules and categories (e.g., [3, Chapters 2–3], [4, Chapters 10–12, Appendix II], [10, Chapter 3])
  - (a) Vector spaces and linear algebra
  - (b) Modules: simple, indecomposable, free, isomorphism theorems
  - (c) Basic category theory: universal definitions, functors
  - (d) Exact sequences, sections, retractions; and connections to free and projective modules
  - (e) Products and coproducts of modules: $\mathrm{Hom}(\cdot,\cdot)$ as a functor and exactness properties, and $\mathrm{Hom}(\bigoplus_i M_i, \prod_j N_j)$
  - (f) Localization, tensor products, extension of scalars, flat modules; and connections with coproducts, pullbacks, pushouts
  - (g) Finitely generated modules over a PID: general structure theory, linear operators and vector spaces as $k[x]$-modules, applications to canonical forms

Date: July 10, 2017.

(4) Fields (e.g., [4, Chapters 13–14], [10, Chapters 5–6], [15, Chapter 11])
   (a) Finite and algebraic field extensions, splitting fields, composite extensions
   (b) Separability and normality, extending embeddings, existence and uniqueness of splitting fields, algebraic closures and uniqueness
   (c) Compass and straightedge constructions, cyclotomic polynomials
   (d) Fundamental theorem of Galois theory (for finite extensions), normality in subextensions, composites and liftings of Galois extensions
   (e) Examples: cyclotomic, biquadratic, splitting field of $x^n - a$ over $\mathbb{Q}$
   (f) Finite fields, irreducibles over finite fields, finite abelian groups are Galois groups, prime cyclotomic fields and primitive elements
   (g) Solvability by radicals: insolvability of the quintic, the general polynomial of degree n, Artin's theorem on characters, norm and trace, Hilbert's Theorem 90

2. Possible supplemental topics

In consultation with your oral exam committee, choose one of these supplemental topics to study (or suggest your own!) The bulk of your exam will be spent verifying that you have a solid command of the core topics, and a small amount of time will be devoted to seeing how well you did learning some new, more advanced topic algebra.

(1) Advanced linear algebra: multilinear algebra, tensor algebras, exterior algebras, duality, pairings, self-adjoint, unitary, and orthogonal transformations, spectral theorem for normal operators. [1, Chapter I], [10, Chapter XV-XVI]
(2) Quadratic forms: orthogonal groups, diagonalization, bilinear forms (symmetric, alternating), Clifford algebras [4, Chapter 11] [8, Chapter I], [12, Chapters IV-V]
(3) Algebraic geometry: affine algebraic sets and correspondence between varieties and prime ideals, Hilbert's Nullstellensatz, Krull dimension, [6, Chapter I], [3, Chapters 1, 7, 8]
(4) Representation theory of finite groups, [4, Chapters 18-19], [13, Chapters 1-3]
(5) Integral extensions: integrality, extensions of prime ideals, integral closure, Dedekind domains [3, Chapters 5,9], [11, Chapter 1]
(6) Local fields (valuation theory) [2, Chapters 1–3], [12, Chapter 1], [14, Chapters I–II], [16, Chapters 1-3]
(7) Noncommutative rings (Wedderburn–Artin theory) [5, Chapter 1], [9, Chapter 1]

References

[1] Emil Artin, *Geometric algebra*, John Wiley & Sons, 2011.
[2] Emil Artin, *Algebraic numbers and algebraic functions*, vol. 358, American Mathematical Soc., 1967.
[3] Michael Francis Atiyah and Ian Grant Macdonald, *Introduction to commutative Algebra*, Reading, Addison-Wesley, 1969.
[4] David Dummit and Richard Foote, *Abstract algebra*, 3rd ed., Wiley, Englewood Cliffs, NJ, 2004.
[5] Benson Farb and R. Keitn Dennis, *Noncommutative algebra*, Graduate Texts in Math., vol. 144, 1993.
[6] William Fulton, *Algebraic curves: an introduction to algebraic geometry*, Addison-Wesley, 1989.
[7] Larry Grove, *Classical groups and geometric algebra*, vol. 39, American Mathematical Soc., 2002.
[8] Tsit-Yuen Lam, *Introduction to quadratic forms over fields*, American Mathematical Soc., 2005.
[9] Tsit-Yuen Lam, *A first course in noncommutative rings*, Graduate Texts in Math., Springer, 2013.
[10] Serge Lang, *Algebra*, Graduate Texts in Math., vol. 211, 2002.
[11] Serge Lang, *Algebraic number theory*, Graduate Texts in Math., vol. 110, Springer, 2013.
[12] Onorato Timothy O'Meara, *Introduction to quadratic forms*, vol. 117, Springer, 2013.

[13] Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Math., vol. 42, Springer, 2012.
[14] Jean-Pierre Serre, *Local fields*, Graduate Texts in Math., vol. 67, Springer, 2013.
[15] Mark Steinberger, *Algebra*, Brooks-Cole, 1994.
[16] Edwin Weiss, *Algebraic number theory*, Courier, 1998.