Math 101 Syllabus

Standard Text: Dummit and Foote: Abstract Algebra, Chapters 4, 5, 10, 11, 12

1. [4 days] Basic Linear Algebra:
   (a) (Assumed) Linear independence, span, basis, dimension, independent sets extend to a basis, generating sets can be pared down to a basis.
   (b) Coordinates and matrix of a linear transformation relative to a basis, change of basis. Examples: projection onto a hyperplane, rotations.
   (c) Row reduction, echelon form, and consequences: free variables, pivot variables, kernel and image, rank-nullity theorem for $T : \mathbb{R}^n \to \mathbb{R}^m$ (via free and pivot variables), elementary row operations and invertibility. Parallel comments for column operations. Given $A \in M_{m\times n}(F)$, discuss representative of cosets $GL_m(F)A$, $AGL_n(F)$, and $GL_m(F)AGL_n(F)$, the last as precursor to Smith normal form.
   (d) Rank - Nullity (vector space form)
   (e) Foreshadow Smith normal form by considering $A \in M_{m\times n}(\mathbb{Z})$ and row and column operations (over $\mathbb{Z}$) to produce the nice representative in $GL_m(\mathbb{Z})AGL_n(\mathbb{Z})$ (when $m = n$, $\text{diag}(d_1, \dots, d_n)$ with $d_i \in \mathbb{Z}$ and $d_i \mid d_{i+1}$, $1 \le i \le n-1$). Example: structure of $\mathbb{Z}^n/K$ where K is a subgroup generated by a collection of vectors. Interpret as linear map and use two sided equivalence to produce a new basis so that $\mathbb{Z}^n/K \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$ ($d_i \mid d_{i+1}$) [foreshadowing invariant factor theorem].

2. [4 days] Modules: basic properties.
   (a) Definitions, examples (vector spaces, abelian groups, $T : V \to V$ linear map to $k[x]$-module structure on V. Notion of a k-algebra ($M_n(k)$, $k[x]$, $End_k(V)$, $k[G]$) and UMP: given any k-algebra A and $a \in A$ there is a unique k-algebra map $k[x] \to A$ taking $x \mapsto a$.
   (b) Direct sums of modules (external and internal); spin off internal direct product of groups. Discuss product and direct sum of vector spaces, mapping properties. Define product and coproduct of modules and their construction. Show $\text{Hom}_R(N, \prod M_\alpha) \cong \prod_\alpha \text{Hom}_R(N, M_\alpha)$, $\text{Hom}_R(\coprod_\alpha M_\alpha, N) \cong \prod_\alpha \text{Hom}_R(M_\alpha, N)$ and $End(k^n) = \text{Hom}(k^n, k^n) \cong M_n(End_k(k)) \cong M_n(k)$

3. [3 days] Exact sequences of modules; split exact sequences via sections or retractions (existence of section equivalent to existence of a retraction). Free modules and their construction; Short exact sequences with $0 \to N \to M \to F \to 0$ with F free split. Any R-module is the quotient of a free R-module (review isomorphism theorems if needed). Localization of modules, connection to exactness, action on direct sums; application: rank of a module over an integral domain is the dimension of the localization over the field of fractions.

4. [6 days] PIDs; Finitely generated modules over PIDs, invariant factor and elementary divisor theorems, applications to rational and Jordan canonical forms. Diagonalizability.

5. [1 day] Dual Modules (duality and free modules)

6. [2 days] Sesquilinear forms. Unitary, Hermitian operators, unitary diagonalization. Real symmetric matrices and spectral theorem.

7. [8 days] Group actions, G-set structure theorem, class equation, p-groups symmetric group, conjugacy classes in S_n, Sylow theorems, semidirect products and split extensions, classifying groups of small orders.

Optional topics:

1. [2 days] (optional) Bilinear forms, isometry groups, connections to dual spaces.