# Absorbing Markov Chains

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# More Examples of Markov Chains

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability $a$ that a person will change the answer from yes to no when transmitting it to the next person and a probability $b$ that he or she will change it from no to yes. We choose as states the message, either yes or no. The transition matrix is then

$$P = \begin{array}{c} \\ \text{yes} \\ \text{no} \end{array} \begin{array}{c} \begin{array}{cc} \text{yes} & \text{no} \end{array} \\ \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array} .$$

The initial state represents the President's choice.

In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. We form a Markov chain with transition matrix

$$P = \begin{array}{c} \\ \text{H} \\ \text{Y} \\ \text{D} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \text{H} & \text{Y} & \text{D} \end{array} \\ \begin{pmatrix} .8 & .2 & 0 \\ .3 & .4 & .3 \\ .2 & .1 & .7 \end{pmatrix} \end{array} .$$

# Absorbing Markov Chains

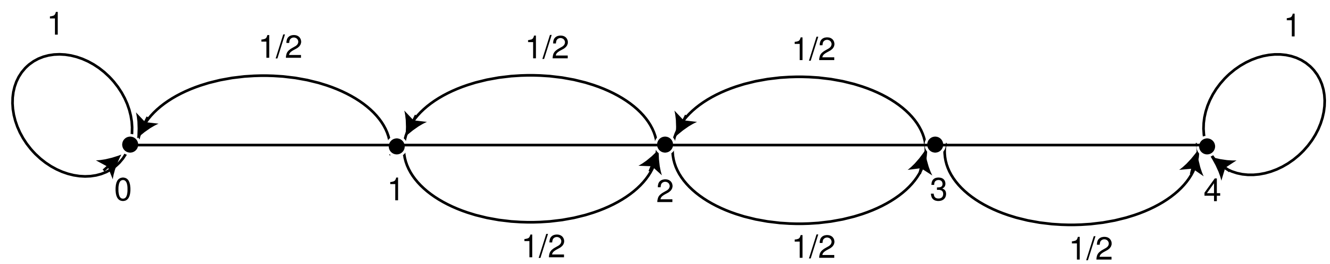
- A state $s_i$ of a Markov chain is called *absorbing* if it is impossible to leave it (i.e., $p_{ii} = 1$).

- A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

- In an absorbing Markov chain, a state which is not absorbing is called *transient*.

# Example: Drunkard's Walk

A man walks along a four-block stretch of Park Avenue. If he is at corner 1, 2, or 3, then he walks to the left or right with equal probability. He continues until he reaches corner 4, which is a bar, or corner 0, which is his home. If he reaches either home or the bar, he stays there.



## The Transition Matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} .$$

# Canonical Form

- For an absorbing Markov chain, renumber the states so that the transient states come first.

- If there are $r$ absorbing states and $t$ transient states, the transition matrix will have the following *canonical form*

$$P = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{cc} \text{TR.} & \text{ABS.} \\ \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) \end{array}$$

- The first $t$ states are transient and the last $r$ states are absorbing.

- $I$ is an $r$-by-$r$ identity matrix, $0$ is an $r$-by-$t$ zero matrix, $R$ is a nonzero $t$-by-$r$ matrix, and $Q$ is an $t$-by-$t$ matrix.

- Recall that the entry $p_{ij}^{(n)}$ of the matrix $P^n$ is the probability of being in the state $s_j$ after $n$ steps, when the chain is started in state $s_i$.

- $P^n$ is of the form

$$P^n = \begin{array}{c c} & \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \begin{array}{c} \text{TR.} \\ \text{ABS.} \end{array} & \left( \begin{array}{c|c} Q^n & ? \\ \hline 0 & I \end{array} \right) \end{array}$$

# Probability of Absorption

**Theorem.** *In an absorbing Markov chain, the probability that the process will be absorbed is 1 (i.e.,* $Q^n \to 0$ *as* $n \to \infty$*).*

## The Fundamental Matrix

**Theorem.** *For an absorbing Markov chain the matrix* $I - Q$ *has an inverse* $N$ *and* $N = I + Q + Q^2 + \cdots$ *.*

For an absorbing Markov chain $P$, the matrix $N = (I - Q)^{-1}$ is called the *fundamental matrix* for $P$. The entry $n_{ij}$ of $N$ gives the expected number of times that the process is in the transient state $s_j$ if it is started in the transient state $s_i$.

# Drunkard's Walk example

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ 0 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \left(\begin{array}{ccc|cc} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} .$$

$$Q = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} ,$$

and

$$I - Q = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix} .$$

$$Q = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \; ,$$

and

$$I - Q = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix} \; .$$

$$N = (I - Q)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \end{array} \; .$$

## Time to Absorption

**Theorem.** *Let $t_i$ be the expected number of steps before the chain is absorbed, given that the chain starts in state $s_i$, and let $t$ be the column vector whose $i$th entry is $t_i$. Then*

$$t = Nc \ ,$$

*where $c$ is a column vector all of whose entries are 1.*

# Absorption Probabilities

Let $b_{ij}$ be the probability that an absorbing chain will be absorbed in the absorbing state $s_j$ if it starts in the transient state $s_i$. Let $B$ be the matrix with entries $b_{ij}$. Then $B$ is an $t$-by-$r$ matrix, and

$$B = NR \ ,$$

where $N$ is the fundamental matrix and $R$ is as in the canonical form.

# In the Drunkard's Walk example

$$N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \end{array} .$$

$$\begin{aligned} t = Nc \quad &= \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} . \end{aligned}$$

$$B = NR \quad = \quad \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$= \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 4 \end{array} \\ \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \end{array} .$$