

# The Fractal Nature of Schottky Groups

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Based on the work of David Mumford et al. in *Indra's Pearls*

## Schottky Groups

In this poster we study the limit sets of Schottky groups composed of two Möbius maps  $a$  and  $b$  and their inverses  $A$  and  $B$ . When composed together to create infinite "words" such as  $aBBaBAbAbabaB \dots$  these maps form limit sets whose nature can be determined based on certain properties of the groups. The images seen below are represent approximations of the limit sets of Schottky groups with carefully chosen parameters.

## Möbius maps

A Möbius map  $T$  takes

$$z \rightarrow \frac{az + b}{cz + d},$$

where  $z, a, b, c, d \in \mathbb{C}$ . Let  $a$  be one such map and let  $A$  be its inverse, that is,  $aA(z) = Aa(z) = z$ . We can think of  $a$  as the  $2 \times 2$  matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and indeed the composition of two Möbius maps is given by the product of their matrix representations. The maps we are interested in are the maps that pair two circles together: given two circles  $x$  and  $y$ , the map takes the outside of  $x$  to the inside of  $y$ . Taking two Möbius maps  $a$  and  $b$ , whose circle pairs do not overlap, we calculate their inverses as  $A$  and  $B$ . Repeatedly applying these maps to a point maps the whole complex plane into one of these four circles. The points that result from all infinite combinations of  $a, b, A$ , and  $B$  are called the limit set of the generators  $a$  and  $b$ , and the set of all the combinations themselves is called a Schottky group.

The images on this page are all the limit sets of specially chosen Schottky groups. We must go over some complex arithmetic to understand how they are formed. In complex arithmetic, division by 0 is allowed, with

$$\pm \frac{z}{0} = \infty$$

for  $z \in \mathbb{C}, z \neq 0$ .  $\infty$  is further defined by  $z \pm \infty = \infty$ , i.e.  $-\infty$  does not exist. Thus a Möbius map can take any point in the complex plane to any other point, with  $\infty$  as one "point". By setting  $a, b, c, d$  as desired, a Möbius map can represent any translation  $T(z) = z + a$ , any rotation  $T(z) = kz$  with  $|k| = 1$ , and any scaling  $T(z) = kz, |k| > 1$ . It turns out that any Möbius map  $T$  is conjugate to one of these three transformations, where  $\tilde{T} = RTR^{-1}$  is the conjugate of  $T$  under some transformation  $R$ .

Every Möbius map can be classified as one of three types: loxodromic, parabolic, or elliptic. A parabolic map is conjugate to a translation, an elliptic map is conjugate to a rotation, and a loxodromic map is conjugate to a scaling. Parabolic maps, being translations, have the special property that they have only one fixed point. A fixed point of a Möbius map is any point  $z \in \mathbb{C}$  such that  $T(z) = z$ . If  $T = kz$ , then the fixed points of  $T$  are 0 and  $\infty$ , since  $T(0) = 0$  and  $T(\infty) = \infty$ . The fixed points of a transformation are given by the solutions to the equation

$$z = \frac{az + b}{cz + d}.$$

A parabolic transformation has only the fixed point  $z = \infty$ . The above equation has only one solution only when  $a + d = \text{Tr } T = \pm 2$ , where  $\text{Tr } T$  is the trace of  $T$ . If we let  $a$  and  $b$  be parabolic and let their circle pairs be tangent, the composite map  $abAB$  has only one fixed point at the tangent between the circles  $a$  and  $b$ . If we specify in addition that  $\text{Tr } abAB = -2$ , then the limit set of the Schottky group generated by  $a$  and  $b$  is a continuous curve.

The limit sets of Schottky groups can be found by using a depth-first search (DFS) algorithm. Of course we cannot explore all the infinite compositions of  $a, b, A$ , and  $B$ , but if we set a high enough depth then the result is the limit set as far as we can tell. The images below were generated using the DFS algorithm outlined on page 148 of *Indra's Pearls*.

## Apollonian Gasket

Somewhat surprisingly, the Apollonian gasket can be generated by finding the limit set of a parabolic Schottky group. We generated figures 1 and 2 by using the DFS algorithm on the generators

$$a = \begin{pmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{pmatrix}, \quad b = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

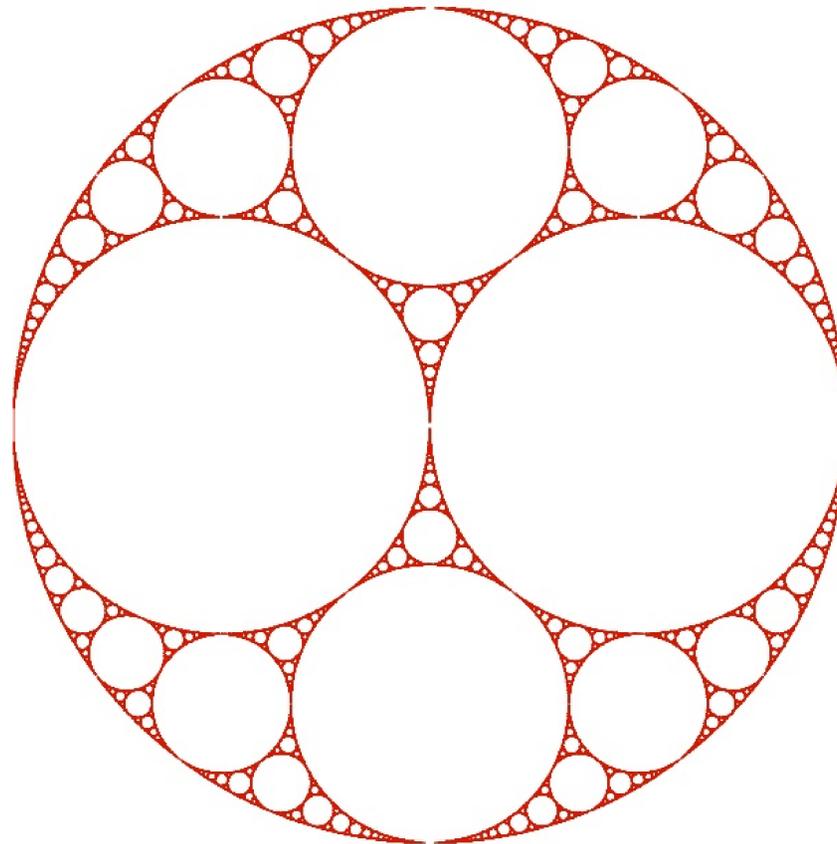


Figure 1: The Apollonian gasket generated by a Schottky group.

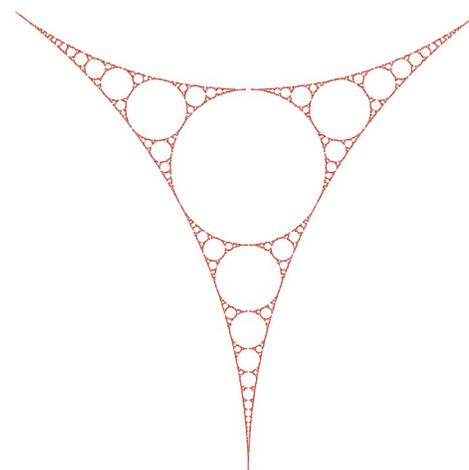


Figure 2: A close-up view of the gasket.

## The significance of trace

The generators  $a$  and  $b$  can be determined completely by  $\text{Tr } a$ ,  $\text{Tr } b$ , and  $\text{Tr } ab$ . We can solve for  $\text{Tr } ab$  from the first two, so by setting  $\text{Tr } a$  and  $\text{Tr } b$  we can create completely new generators. Figures 3 and 4 were generated by  $\text{Tr } a = 1.87 + 0.5i$ ,  $\text{Tr } b = 1.87 - 0.5i$  and  $\text{Tr } a = 1.87 + 0.1i$ ,  $\text{Tr } b = 1.87 - 0.1i$ . Since they have complex traces these groups are loxodromic. Smaller imaginary parts lead to tighter spirals, as shown by the difference between figure 3 and figure 4.

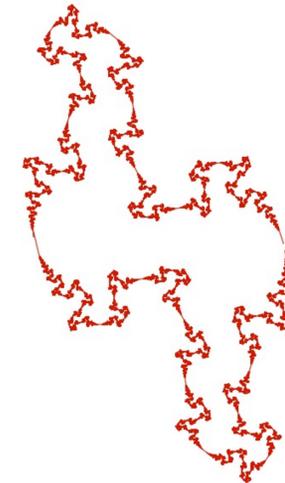


Figure 3: Jordan curve generated by  $\text{Tr } a = 1.87 + 0.5i$ ,  $\text{Tr } b = 1.87 - 0.5i$ .

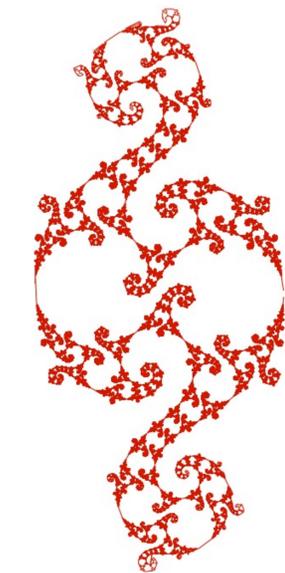


Figure 4: Jordan curve generated by  $\text{Tr } a = 1.87 + 0.1i$ ,  $\text{Tr } b = 1.87 - 0.1i$ .

## References

Mumford, David, Caroline Series, and David Wright. *Indra's Pearls: The Vision of Felix Klein*. Cambridge, U.K.: Cambridge University Press, 2002. Print.