The Fundamental Theorem of Calculus
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We are about to discuss a theorem that relates derivatives and definite integrals. It is so important in the study of calculus that it is called the Fundamental Theorem of Calculus. It also gives us a practical way to evaluate many definite integrals without resorting to the limit definition. The theorem has two main parts that we will state separately as Part I and Part II.

Fundamental Theorem of Calculus (Part I-antiderivative): Suppose that \( f \) is a continuous function on the interval \( I \) containing the point \( a \). Define the function \( F \) on \( I \) by the integral formula

\[
F(x) = \int_a^x f(t) \, dt
\]

Then \( F \) is differentiable on \( I \) and \( F'(x) = f(x) \). That is, \( F \) is an antiderivative of \( f \) on \( I \).

Fundamental Theorem of Calculus (Part II-evaluation): If \( G(X) \) is any antiderivative of \( f \) on \( I \) (that is, \( G'(x) = f(x) \) on \( I \)), then for any \( b \) in \( I \),

\[
\int_a^b f(x) \, dx = G(b) - G(a)
\]

This theorem is truly remarkable. Leibniz seems to have been the first one to recognize its generality and significance. Let’s look at some examples so that we can gain a better understanding of what the theorem says, and then we will outline a proof.

**Example 1:** To compute \( \int_0^1 (x+1) \, dx \), we need only find an antiderivative of \( x+1 \), namely, \( x^2/2 + x \). Then we evaluate this antiderivative at 1 and subtract its value at 0. Thus,

\[
\int_0^1 (x+1) \, dx = \left( \frac{1}{2} + 1 \right) - \left( 0 \right) = \frac{3}{2}.
\]

We normally use a vertical bar to indicate evaluation of the antiderivative at the endpoints of the interval. That is,

\[
G(x)|_a^b = G(b) - G(a)
\]

**Example 2:**

\[
\int_0^1 x^2 \, dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.
\]

**Example 3:**

\[
\int_0^{\pi/4} \sin x \, dx = -\cos x|_0^{\pi/4} = -1/\sqrt{2} - (-1) = 1 - 1/\sqrt{2}.
\]

**Example 4:**

\[
\int_0^{\pi/4} \sec^2 x \, dx = \tan x|_0^{\pi/4} = 1 - 0 = 1.
\]

We can also illustrate Part I of the Fundamental Theorem.

**Example 5:**

\[
\frac{d}{dx} \int_1^x t^2 \, dt = x^2.
\]

**Example 6:**

\[
\frac{d}{dx} \int_1^{x^2} t^3 \, dt = (x^2)^3 \cdot 2x \text{ where we first have used the Fundamental Theorem and then the chain rule to complete the calculation of the derivative.}
\]

**Example 7:** Consider \( \frac{d}{dx} \int_{x^2}^{x^3} e^{-t^2} \, dt \). We first have to put the integral in the correct form so that we can use the Fundamental Theorem:

\[
\frac{d}{dx} \int_{x^2}^{x^3} e^{-t^2} \, dt = \frac{d}{dx} \left( \int_0^{x^3} e^{-t^2} \, dt + \int_0^{x^2} e^{-t^2} \, dt \right) = \frac{d}{dx} \left( -\int_0^{x^2} e^{-t^2} \, dt + \int_0^{x^3} e^{-t^2} \, dt \right) = -e^{-x^4}(2x) + e^{-x^6}(3x^2)
\]
Now that we have gained some experience with the Fundamental Theorem through examples, let’s look at a sketch of a proof in a special case.

**Proof of the Fundamental Theorem (Part I):** Fix \( x \) in \( I \). Given that \( F(x) = \int_{a}^{x} f(t) \, dt \), we need to evaluate the limit

\[
\lim_{h \to 0} \frac{F(x + h) - F(x)}{h}
\]

But look at the sketch below. Notice that \( F(x) \) is the area under the graph of \( f \) and above the interval \([a, x]\), while \( F(x + h) \) is the area under the graph of \( f \) and above the interval \([a, x + h]\). Thus, \( F(x + h) - F(x) \) is the area under the graph of \( f \) and above the interval \([x, x + h]\).

But for small values of \( h \), this area is approximately equal to the area of the rectangle of height \( f(x) \) on the same base; its area is length times width, or \( h \cdot f(x) \). Thus, for small \( h \), the difference quotient is approximately equal to \( \frac{h \cdot f(x)}{h} = f(x) \). In other words,

\[
F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x)
\]

thereby completing the proof of Part I.

**Proof of the Fundamental Theorem (Part II):** From Part I, we have that \( F(x) = \int_{a}^{x} f(t) \, dt \) is an antiderivative of \( f \). If \( G \) is another antiderivative, then we know from a previous result that they must differ by a constant. That is, \( G(x) = F(x) + C \). Now, we know that \( F(a) = \int_{a}^{a} f(t) \, dt = 0 \). Thus, we can determine the value of \( C \): \( G(a) = F(a) + C = 0 + C = C \). Hence, \( G(x) = F(x) + G(a) \), or \( F(x) = G(x) - G(a) \). So, if \( b \) is any point in \( I \), we have \( G(b) - G(a) = F(b) = \int_{a}^{b} f(t) \, dt \), which is what we wanted to prove.

**Another Proof of the Fundamental Theorem of Calculus**

**Theorem statement:** If \( G(x) \) is any antiderivative of \( f \) on \( I \) (that is, \( G'(x) = f(x) \) on \( I \)), then for any \( b \) in \( I \).

\[
\int_{a}^{b} f(x) \, dx = G(b) - G(a)
\]

We are going to prove this result by an application of Euler’s Method which we studied earlier. Suppose we consider the Initial Value Problem

**IVP:** \( y' = f(x), y(a) = 0, a \leq x \leq b \), where \( a, b \) are in \( I \).
and we want to find \( y(b) \). Then because both \( y \) and \( G \) are antiderivatives of \( f \) on \([a, b]\), \( y(x) = G(x) + C \) for some constant \( C \) on \([a, b]\). Then \( 0 = y(a) = G(a) + C \) implies \( C = -G(a) \) and hence \( y(b) = G(b) - G(a) \). Now, we will use Euler’s method to approximate \( y(b) \).

Suppose we use an integral number \( n \) of steps where each step has size \( \frac{b-a}{n} \). Then, starting at the point \((a, 0)\) where the slope is \( y'(a) = f(a) \), we generate the following points:

<table>
<thead>
<tr>
<th>Point ((x, y))</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a, 0))</td>
<td>(f(a))</td>
</tr>
<tr>
<td>((a + h, f(a) h))</td>
<td>(f(a + h))</td>
</tr>
<tr>
<td>((a + 2h, f(a) h + f(a + h) h))</td>
<td>(f(a + 2h))</td>
</tr>
<tr>
<td>etc.</td>
<td>etc.</td>
</tr>
</tbody>
</table>

The endpoint at \( x = b \) has \( y \)-coordinate

\[
\sum_{i=0}^{n-1} f(a + ih) h
\]

The above sum is the Euler method approximate value of \( y(b) \) which converges to \( y(b) \) as \( h \to 0 \). But note that it is also a Riemann sum for the definite integral from \( a \) to \( b \) of \( f \), and the Riemann sum converges to the value of the integral as \( h \to 0 \). Thus, because the limit of the sum is unique, we have

\[
y(b) = \int_{a}^{b} f(x) \, dx
\]

and from the result \( y(b) = G(b) - G(a) \) in the first paragraph of the proof, we see that the proof is complete.

**Exercises:** Problems Check what you have learned!

**Videos:** Tutorial Solutions See problems worked out!