We have already seen that the differential equation \( \frac{dy}{dx} = ky \), where \( k \) is a constant, has solution \( y = y_0 e^{kx} \). We have solved this equation in three ways: by guess-and-check in Section 3.1, and by algebraic manipulation and integration in Section 3.2. The differential equation, representing exponential growth or decay, is also an example of a separable differential equation, which we solved as such in Section 3.1.

As introduced in Section 3.1, a first-order differential equation in \( x \) and \( y \) is called separable if it is of the form

\[
\frac{dy}{dx} = g(x)h(y)
\]

where \( y = f(x) \). That is, when the equation is written in terms of differentials, the \( x \)'s and \( dx \)'s can be put on one side of the equation and the \( y \)'s and \( dy \)'s on the other in such a way that we can solve the equation by integrating both sides:

\[
\int \frac{1}{h(y)} \, dy = \int g(x) \, dx
\]

This procedure to solve the differential equation is called the method of separation of variables.

**Example 1:** As a review, let's again solve the equation \( \frac{dy}{dx} = ky \) by the method of separation of variables. The method begins by rewriting the equation using differentials. First, we separate the \( y \)'s and \( dy \)'s from the \( x \)'s and \( dx \)'s, and then we integrate both sides of the rewritten equation, and solve for \( y \):

\[
\int \frac{1}{y} \, dy = k \, dx
\]

\[
\ln |y| = kx + C
\]

From this point on, we do exactly what we did before: we solve for \( y \) by exponentiating both sides:

\[
|y| = e^{kx+C}
\]

\[
y = \pm e^C e^{kx} = y_0 e^{kx}
\]

**Justification for the Method of Separation of Variables:** But why is the method of separation of variables valid? After all, on the left side of the separated equation we are integrating with respect to \( y \), and on the right side with respect to \( x \). Using differentials facilitates the method and is a reflection of the genius of Leibniz who believed that the notation should be chosen to motivate the correct answer. However, we have just described a subtlety that we don’t want to slide over. The method does indeed give the correct answer, but we must prove it. *Proof by notation* will not suffice.

In fact, we need to show that given the equation

\[
\frac{dy}{dx} = g(x)h(y)
\]
\[
\frac{dy}{dx} = g(x)h(y)
\]
the antiderivative of \(\frac{1}{h(y)}\) as a function of \(y\) equals the antiderivative of \(g(x)\) as a function of \(x\).

The function \(y = f(x)\) is a solution of the above equation implies that

\[
\begin{align*}
    f'(x) &= g(x)h(f(x)) \\
    \frac{f'(x)}{h(f(x))} &= g(x)
\end{align*}
\]

Let \(H(y)\) be any antiderivative of \(1/h(y)\); so \(H'(y) = 1/h(y)\). Then applying the chain rule yields

\[
\frac{d}{dx} H(f(x)) = H'(f(x)) f'(x)
\]
\[
= f'(x) \frac{1}{h(f(x))}
\]
\[
= g(x)
\]

Thus, the solution \(y = f(x)\) satisfies the equation

\[
H(f(x)) = \int g(x) \, dx
\]

However, this is just the result of the method of separation of variables, which is to rewrite the differential equation as

\[
\frac{1}{h(y)} \, dy = g(x) \, dx
\]

and to integrate both sides (the left side with respect to \(y\) and the right with respect to \(x\)) obtaining an equation of the form

\[
H(y) = \int g(x) \, dx
\]

Then this equation implicitly defines the solution \(y = f(x)\), as desired.

**More Examples of the Method of Separation of Variables:** In the rest of the section, we will consider additional examples of solving separable differential equations.

**Example 2:** We can use the method of separation of variables to solve the differential equation \(\frac{dy}{dx} = \frac{x}{y}\).

\[
\begin{align*}
    y \, dy &= x \, dx \\
    \int y \, dy &= \int x \, dx \\
    \frac{y^2}{2} &= \frac{x^2}{2} + C \\
    y^2 - x^2 &= C_1
\end{align*}
\]
The solution curves are hyperbolas. We can’t really go any further unless we knew, say, a point that the solution curve passed through.

Example 3: Solve the IVP \( \frac{dy}{dx} = x^2 y^3; y(3) = 1 \). Separating the variables and integrating, we get:

\[
\frac{1}{y^3} dy = x^2 dx
\]

\[
\int \frac{1}{y^3} dy = \int x^2 dx
\]

\[
\frac{1}{2y^2} = \frac{x^3}{3} + C
\]

Here are some solution curves:

From \( y(3) = 1 \), we find the particular solution:

\[
-\frac{1}{2} = \frac{27}{3} + C
\]

\[
C = -\frac{19}{2}
\]

\[
-\frac{1}{2y^2} = \frac{x^3}{3} - \frac{19}{2}
\]

\[
y^2 = \frac{1}{19 - \frac{2x^3}{3}}
\]

\[
y = \sqrt{19 - \frac{2x^3}{3}}
\]

Note that we know that \( y \) is the positive square root because we have the initial condition \( y(3) = 1 \). Here is the particular solution:
Example 4: Solve $\frac{dy}{dx} = \frac{2y}{x}$.

\[
\frac{1}{y} \, dy = \frac{2}{x} \, dx
\]
\[
\int \frac{1}{y} \, dy = \int \frac{2}{x} \, dx
\]
\[
\ln |y| = 2 \ln |x| + C
\]
\[
|y| = x^2 e^C
\]
\[
y = C_1 x^2
\]

The solution curves are a family of parabolas.

Example 5: Solve $\frac{dy}{dx} = -\frac{x}{2y}$.

\[
2y \, dy = -x \, dx
\]
\[
\int 2y \, dy = - \int x \, dx
\]
\[
y^2 = -\frac{x^2}{2} + C
\]
\[
2y^2 + x^2 = C_1
\]

The solutions are a family of ellipses:
Example 6: We can also solve Torricelli’s equation by the method of separation of variables. We found in Section 2.18 that the equation is of the form \( y' = k\sqrt{y} \), where \( k \) is a constant. Then we have:

\[
\int y^{-\frac{1}{2}} \, dy = k \int dx
\]

\[
\int y^{-\frac{1}{2}} \, dy = \int k \, dx
\]

\[
2 y^{1/2} = kx + C
\]

\[
y^{1/2} = \frac{1}{2} kx + C_1
\]

\[
y = \left( \frac{1}{2} kx + C_1 \right)^2
\]

This is the form of the general solution that we explored in the case study of the previous section.

Exercises: Problems Check what you have learned!

Videos: Tutorial Solutions See problems worked out!