The Peter-Weyl Theorem for Compact Groups

The following notes are from a series of lectures I gave at Dartmouth College in the summer of 1989. The general outline is provided by an introductory section in [1] but with considerable detail added by myself. The mistakes of course are mine.

Dana P. Williams
Hanover, June 1991

§1 Preliminaries.
We begin with some warm-up exercises on locally compact groups; a.k.a., a long series of definitions! At this point, $G$ is meant to be an arbitrary locally compact group.

Definition 1: A (unitary) representation of $G$ is a continuous homomorphism $\pi$ from $G$ to the unitary group $\mathcal{U}(\mathcal{H}_\pi)$ on a (complex) Hilbert space $\mathcal{H}_\pi$ equipped with the strong operator topology.

Remark 2: The condition that $\pi$ be continuous merely means that $g \mapsto \pi(g)\xi$ is continuous from $G$ to $\mathcal{H}_\pi$ for each $\xi \in \mathcal{H}_\pi$. There are many equivalent conditions: the weakest I'm aware of is to insist that $g \mapsto \langle \pi(g)\xi, \eta \rangle$ be Borel for each $\xi, \eta \in \mathcal{H}_\pi$.

Example 3: Let $\mathcal{H} = L^2(G)$. The left regular representation $\lambda : G \to \mathcal{B}(\mathcal{H})$ is given by $\lambda(g)f(t) = f(g^{-1}t)$.

Definition 4: Two representations $\pi_1$ and $\pi_2$ are said to be (unitarily) equivalent if there is a unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $\pi_1(g) = U\pi_2(g)U^*$ for all $g \in G$. In this event, we write $\pi_1 \cong \pi_2$ and let $[\pi]$ denote the (unitary) equivalence class of $\pi$.

If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a representation of $G$, then we'll write $d_\pi$ for the dimension (in $\{0, 1, 2, \ldots, \infty\}$) of $\mathcal{H}_\pi$. Fortunately, whenever $\pi_1 \cong \pi_2$, then it is clear that $d_{\pi_1} = d_{\pi_2}$. Thus we will often write $d_{[\pi]}$ to denote the dimension of each representation in the same equivalence class as $\pi$.

Definition 5: A non-zero representation $\pi$ is called irreducible if $\mathcal{H}_\pi$ has no non-trivial closed invariant subspaces.

Remark 6: If $d_\pi < \infty$, then the word “closed” in redundant in the above definition.

Definition 7: The symbol $\hat{G}$ is used to denote the collection of equivalence classes of irreducible representations of $G$.

Example 8: If $G$ is abelian, then every irreducible representation is one-dimensional. In particular, $\hat{G}$ coincides with the character group of $G$.  

1
**Definition 9:** If $\pi$ and $\eta$ are representations of $G$, then $\pi \oplus \eta$ denotes the representation on $\mathcal{H}_\pi \oplus \mathcal{H}_\eta$ defined by

$$\pi \oplus \eta(g)(\xi, \zeta) = (\pi(g)\xi, \eta(g)\zeta).$$

If $n \in \mathbb{Z}^+ \cup \{\infty\}$, then $n \cdot \pi = \oplus_{i=1}^{n} \pi$.

**Definition 10:** If $\pi$ is a representation of $G$, then

$$\pi(G)' = \{ A \in \mathcal{B}(\mathcal{H}_\pi) : \pi(g) = \pi(g)A, \text{ for all } g \in G \}.$$

**Remark 11:** Since $\pi$ is unitary, $\pi(G)'$ is a self-adjoint. In fact it is not hard to check that $\pi(G)'$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ which is closed in the weak operator topology.

**Theorem 12:** Suppose that $\pi$ is a representation of $G$. Then the following are equivalent.

1. $\pi$ is irreducible.
2. $\pi(G)' = \mathbb{C}I$.
3. Every non-zero $\xi \in \mathcal{H}_\pi$ is cyclic for $\pi$ (i.e., $\overline{\pi(G)\xi} = \overline{\text{span}\{ \pi(g)\xi : g \in G \}} = \mathcal{H}_\pi$).

**Proof:** Suppose that $\pi$ is irreducible. Let $A \in \pi(G)'$ and suppose for the moment that $A$ is normal: $A^*A = AA^*$. Then the norm closed unital $*$-subalgebra generated by $A$—that is the $C^*$-algebra $C^*(I,A)$—is contained in $\pi(G)'$. Since $A$ is normal, $C^*(I,A)$ is commutative and is isomorphic to $C(\sigma(A))$ by the spectral theorem. If $\sigma(A) \neq \{pt\}$, then we can find nonzero self-adjoint operators $B_1$ and $B_2$ in $C^*(I,A)$ so that $B_1B_2 = B_2B_1 = 0$. Thus $\langle B_1\xi, B_2\eta \rangle = 0$ for all $\xi, \eta \in \mathcal{H}_\pi$. In particular, the closures $V_1$ and $V_2$ of the ranges of $B_1$ and $B_2$, respectively, are closed, non-zero, orthogonal, invariant subspaces for $\pi$. This contradicts the irreducibility of $\pi$; therefore $\sigma(A)$ must be a single point and therefore $A = \alpha I$ for some $\alpha \in \mathbb{C}$. For a general $A \in \pi(G)'$, we apply the above reasoning to $AA^*$ and $A^*A$. Thus, we have $AA^* = \alpha I$ and $A^*A = \beta I$ with $\alpha, \beta > 0$ (since $A \neq 0$). Since $\alpha A = A(A^*A) = \beta A$, we have $\alpha = \beta$ and $A$ is normal. This shows that (1) implies (2).

Since $V = \overline{\pi(G)\xi}$ is a closed, non-zero, invariant subspace, in order to show that (2) implies (3) it will suffice to show that the orthogonal projection $P$ onto any closed, non-zero, invariant subspace $V$ is in $\pi(G)'$. But since $\pi$ is unitary, $V^\perp$ is also invariant. Thus if $\xi, \eta \in \mathcal{H}_\pi$, then

$$\langle P\pi(g)\xi, \eta \rangle = \langle \pi(g)\xi, P\eta \rangle = \langle \pi(g)P\xi + \pi(g)(I-P)\xi, P\eta \rangle,$$

which, since $\pi(g)(I-P)\xi \in V^\perp$ and $P\eta \in V$,
because \( \pi(g)P\xi \in V \).

That (3) implies (1) is clear. \( \square \)
§2 The Peter-Weyl Theorem.

Now we’ll specialize to compact groups $G$. Compact groups are characterized by the fact that any Haar measure $\mu$ on $G$ satisfies $\mu(G) < \infty$. It is customary to normalize Haar measure on a compact group by choosing the unique measure such that $\mu(G) = 1$. Since there is now no possibility of confusion, I’ll simply write

$$\int_G f(g) \, dg$$

for the integral of $f \in L^1(G)$.

Now suppose that $\pi$ is a finite dimensional representation of $G$. If $b = \{e_1, \ldots, e_{d_\pi}\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then for each $g \in G$ the matrix of $\pi(g)$ with respect to $b$ has $ij$th coordinate $\langle \pi(g) e_j, e_i \rangle$. The function $\phi_{ij}(g) = \langle e_i, \pi(g) e_j \rangle$ is called a coordinate function for $\pi$. (It will be convenient to use this convention—even though $\phi_{ij}$ is the complex conjugate of what you might expect. This usage and terminology will be justified, somewhat, by Remark 14 below.) Notice that $\phi_{ij} \in \mathcal{C}(G)$. We’ll write $\mathcal{E}_G$, or just $\mathcal{E}$ when no confusion is likely to arise, for the linear span of the all the functions $\phi(g) = \langle \xi, \pi(g) \eta \rangle$, where $\pi$ ranges over all irreducible representations of $G$ and $\xi$ and $\eta$ range over $\mathcal{H}_\pi$. Since every finite dimensional representation is the direct sum of irreducibles, notice that $\psi(g) = \langle \xi, \pi(g) \eta \rangle$ defines an element of $\mathcal{E}$ for every finite dimensional representation $\pi$—irreducible or not.

Remark 13: When $G$ is abelian, and sometimes in general, the functions in $\mathcal{E}$ are called the trigonometric polynomials. The motivation for this probably comes from the case where $G = T = \mathbb{R}/2\pi\mathbb{Z}$. Then each $f \in \mathcal{E}_T$ has the form

$$f(\theta) = \sum_{n=-k}^{n=k} c_n \exp(\imath n\theta) = \sum_{n=0}^{k} d_n \cos(n\theta) + b_n \sin(n\theta).$$

Remark 14: It is clear that $\mathcal{E}$ is self-adjoint; that is, if $f \in \mathcal{E}$, then so is $f^* \in \mathcal{E}$, where $f^*(g) = \overline{f(g^{-1})}$. This is because matrix coefficients are themselves self-adjoint: $\phi_{ij}^\pi = (\phi_{ij}^\pi)^*$. It is also true that if $f \in \mathcal{E}$, then so is $\tilde{f}$, where $\tilde{f}(g) = f(g^{-1})$. To see this we need to introduce the conjugate Hilbert space $\tilde{\mathcal{H}}$ to a given Hilbert space $\mathcal{H}$. The space $\tilde{\mathcal{H}}$ coincides with $\mathcal{H}$ as an additive group. If $j : \mathcal{H} \to \tilde{\mathcal{H}}$ denotes the identity map, then the Hilbert space structure on $\tilde{\mathcal{H}}$ is given by the formulas $\alpha j(\xi) = j(\overline{\alpha}\xi)$, and
\[ \langle j(\xi), j(\eta) \rangle_{\mathcal{H}} = \langle \eta, \xi \rangle_{\mathcal{K}}. \] If \( \pi \) is a given representation of \( G \) on \( \mathcal{K} \), then we can define a representation \( \hat{\pi} \) on \( \hat{\mathcal{K}} \) in the obvious way: \( \hat{\pi}(g)j(\xi) = j(\pi(g)\xi) \). The assertion follows from the fact that \( \hat{\phi}_{ij}^\pi(g) = \hat{\phi}_{ji}^\pi(g^{-1}) \). Since \( \hat{\phi}_{ij}^\pi(g^{-1}) = \overline{\hat{\phi}_{ij}^\pi(g)} \), it is reasonable to call \( g \mapsto \langle \xi, \pi(g)\eta \rangle \) a coordinate function.

**Definition 15:** Let \( M_n \) be the \( n \times n \) complex matrices. If \( A = (a_{ij}) \in M_n \), then the Hilbert-Schmidt norm of \( A \) is

\[
\| (a_{ij}) \|_{\text{h.s.}} = \sum_{ij} |a_{ij}|^2.
\]

**Remark 16:** If \( A \in M_n \), then \( \| A \|_{\text{h.s.}} = \text{tr}(A^*A) \). In particular, if \( B = U^*AU \) for some unitary matrix \( U \), then \( \| A \|_{\text{h.s.}} = \| B \|_{\text{h.s.}} \). It follows that if \( \pi \) is finite dimensional, then \( \| \pi(g) \|_{\text{h.s.}} \) is well defined and depends only on \([\pi]\). □

Our objective here is to prove the following theorem known as the Peter-Weyl Theorem.

**Theorem 17:** Let \( G \) be a compact group.

1. Every irreducible representation of \( G \) is finite dimensional.
2. If \( \lambda \) is the left-regular representation of \( G \), then
   \[
   \lambda \cong \bigoplus_{[\pi] \in \hat{G}} d_{\pi} \cdot \pi
   \]
3. Given \( g \in G \), there is a \([\pi] \in \hat{G}\) such that \( \pi(g) \neq I \).
4. \( \mathcal{E} \) is dense in \( C(G) \) (and hence in \( L^p(G) \) for \( 1 \leq p < \infty \)).
5. If \( f \in L^2(G) \), then
   \[
   \| f \|_2^2 = \sum_{[\pi] \in \hat{G}} d_{\pi} \cdot \text{tr}(\pi(f)\pi(f)^*) = \sum_{[\pi] \in \hat{G}} d_{\pi} \cdot \| \pi(f) \|_{\text{h.s.}}.
   \]

We’ll need the following preliminary results, some of which may be of interest by themselves.
Lemma 18: Let $\pi_1$ and $\pi_2$ be representations of a locally compact group $G$. If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator satisfying $A\pi_1(g) = \pi_2(g)A$ for all $g \in G$, then $A^*A\pi_1(g) = \pi_1(g)A^*A$ for all $g \in G$.

Proof: One computes as follows:

$$\langle A^*A\pi_1(g)\xi, \eta \rangle = \langle A\pi_1(g)\xi, A\eta \rangle$$
$$= \langle \pi_2(g)A\xi, A\eta \rangle$$
$$= \langle A\xi, \pi_2(g^{-1})A\eta \rangle$$
$$= \langle A\xi, \pi_1(g^{-1})\eta \rangle$$
$$= \langle \pi_1(g)A^*A\xi, \eta \rangle$$

\[\square\]

Lemma 19: Suppose that $\pi_1$ and $\pi_2$ are representations of a locally compact group $G$, that $\pi_1$ is irreducible, and that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is any non-zero bounded linear operator such that $A\pi_1(g) = \pi_2(g)A$ for all $g \in G$. Then $A\mathcal{H}_1$ is a closed invariant subspace for $\pi_2$, and $\pi_1 \cong \pi_2|_{A\mathcal{H}_1}$, where $\pi_2|_{A\mathcal{H}_1}$ is the subrepresentation of $\pi_2$ corresponding to $A\mathcal{H}_1$.

Proof: By Lemma 18, $A^*A \in \pi_1(G)'$. By Theorem 12, $A^*A = \lambda I$. Thus $B = \lambda^{-\frac{1}{2}}A$ is an isometry, and hence a unitary from $\mathcal{H}_1$ onto $A\mathcal{H}_1$. (Note that $A\mathcal{H}_1$ is actually closed in $\mathcal{H}_2$ since $A$ is a multiple of an isometry.) Again Lemma 18 shows that $BB^* \in \pi_2(G)'$. Since $BB^*$ is the orthogonal projection onto $B\mathcal{H}_1 = A\mathcal{H}_1$, it follows that $A\mathcal{H}_1$ is invariant for $\pi_2$ and the assertion follows. \[\square\]

The next result is crucial, and depends heavily on the fact that $G$ is compact.

Proposition 20: Suppose that $\pi_1$ and $\pi_2$ are irreducible representations of a compact group $G$. Fix orthonormal bases $\{e_k^i\}_{k=1}^{d_{\pi_1}}$ for $\mathcal{H}_i$, and put $\phi_{kl}(g) = \langle e_k^i, \pi_i(g)e_l^j \rangle$.

1. If $\pi_1 \ncong \pi_2$, then

$$\int_G \phi_{ij}^1(g)\overline{\phi_{ji}^2(g)} \, dg = 0,$$

for all $1 \leq i, j \leq d_{\pi_1}$, and $1 \leq k, l \leq d_{\pi_2}$.

2. If $\pi_1$ is finite dimensional, then

$$\int_G \phi_{ij}^1(g)\overline{\phi_{kl}^1(g)} \, dg = \delta_{ik}\delta_{jl}\frac{1}{d_{\pi_1}}.$$
for all $1 \leq i,j,k,l \leq d_{\pi_1}$.

**Proof:** Let $B$ be any bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then we can define

$$A = \int_G \pi_2(g) B \pi_1(g^{-1}) ds.$$ 

Then

$$A \pi_1(r) = \int_G \pi_2(g) B \pi_1(g^{-1} r) dg = \int_G \pi_2(rg) B \pi_1(g^{-1}) dg = \pi_2(r) A.$$ 

If $\pi_1 \not\equiv \pi_2$, then $A = 0$ by Lemma 19. Now suppose that $B = B_{ij}$ is the rank-one operator defined by $B_{ij}(\xi) = \langle \xi, e_j^1 e_i^2 \rangle$. Then,

$$0 = \langle Ae_i^1, e_k^2 \rangle = \int_G \langle B_{ij} \pi_1(g^{-1}) e_i^1, \pi_2(g^{-1}) e_k^2 \rangle dg$$

$$= \int_G \langle \pi_1(g^{-1}) e_i^1, e_j^2 \rangle \langle e_i^1, \pi_2(g^{-1}) e_k^2 \rangle dg$$

$$= \int_G \phi_{ij}^1(g) \overline{\phi_{kl}^1(g)} ds.$$ 

This proves (1).

Now assume that $d_{\pi_1} < \infty$. By the above and Lemma 19,

$$A = \int_G \pi_1(g) B \pi_1(g^{-1}) dg = \lambda I,$$ 

for any $B$. Taking traces,

$$\text{tr}(A) = \sum_{k=1}^{d_{\pi_1}} \langle Ae_k^1, e_k^1 \rangle$$

$$= \int_G \sum_{k=1}^{d_{\pi_1}} \langle B \pi_1(g^{-1}) e_k^1, \pi_1(g^{-1}) e_k^1 \rangle dg$$

$$= \int_G \text{tr}(B) dg = \text{tr}(B).$$

Since $\text{tr}(A) = \text{tr}(\lambda I) = \lambda d_{\pi_1}$ and $\text{tr}(B_{ji}) = \delta_{ji}$, we have $\lambda = \delta_{ji} \frac{1}{\pi_{\pi_1}}$ when $B = B_{ji}$. On the other hand,

$$\int_G \phi_{ij}^1(g) \overline{\phi_{kl}^1(g)} dg = \langle Ae_i^1, e_k^1 \rangle = \lambda \langle e_i^1, e_k^1 \rangle = \delta_{ik} \delta_{ji} \frac{1}{d_{\pi_1}}.$$ 

\[ \square \]
Proof of Theorem 17: Let $E_f$ be the subset of $E$ consisting of the collection of matrix coefficients of the form $\phi(g) = \langle \xi, \pi(g) \eta \rangle$ for $\pi$ irreducible and $d_\pi < \infty$. The first part of the proof will consist of showing that it suffices to show that $E_f$ is dense in $C(G)$. Since $C(G)$ is dense in $L^2(G)$, it follows that $L^2(G) = \mathcal{H}_\Lambda$ has an orthonormal basis consisting of normalized matrix coefficients $d_{ij}^{1/2} \phi_{ij}^\pi(g) = d_{ij}^{1/2} \langle \pi^\pi, \pi(g)e_j^\pi \rangle$ (where, of course, $\{ e_1^\pi, \ldots, e_{d_\pi}^\pi \}$ denotes an orthonormal basis for $\mathcal{H}_\pi$).

In fact, if $\pi$ is any irreducible representation of $G$ and if $\{ e_\alpha \}_{\alpha \in A}$ is an orthonormal basis for $\mathcal{H}_\pi$, then it follows from Proposition 20 that $\phi_{\alpha, \beta}^{\pi} \perp \phi_{ij}^\pi$, where $\phi_{\alpha, \beta}^{\pi}(g) = \langle e_\alpha^\pi, \pi(g)e_\beta^\pi \rangle$ and $d_{\rho} < \infty$. Thus we must have $\phi_{\alpha, \beta}^{\pi} = 0$, and (1) follows.

If $d_\pi < \infty$, then Proposition 20 shows that the $d_{\pi}^{2}$ functions $\{ \sqrt{d_\pi} \phi_{ij}^\pi \}$ are an orthonormal basis for a subspace $\mathcal{H}_{[\pi]}$ of $L^2(G)$. Now observe that

$$
\phi_{ij}^\pi(g^{-1}t) = \langle \pi(g)e_i^\pi, \pi(t)e_j^\pi \rangle
$$

$$
= \sum_{k=1}^{d_\pi} \langle \pi(g)e_i^\pi, e_k^\pi \rangle \langle e_k^\pi, \pi(t)e_j^\pi \rangle
$$

$$
= \sum_{k=1}^{d_\pi} \phi_{ik}^\pi(g^{-1}) \phi_{kj}^\pi(t).
$$

Therefore if, for each $1 \leq j \leq d_\pi$, we define $A_j : \mathcal{H}_{\pi} \to \mathcal{H}_{[\pi]}$ by $A_je_j^\pi = \phi_{ij}^\pi$, then

$$
\lambda(g)(A_ie_i^\pi)(t) = \phi_{ij}^\pi(g^{-1}t)
$$

$$
= \sum_{k=1}^{d_\pi} \phi_{ik}^\pi(g^{-1}) \phi_{kj}^\pi(t)
$$

$$
= \sum_{k=1}^{d_\pi} \langle e_i^\pi, \pi(g^{-1})e_k^\pi \rangle A_j e_k^\pi(t)
$$

$$
= A \left( \sum_{k=1}^{d_\pi} \langle \pi(g)e_i^\pi, e_k^\pi \rangle e_k^\pi \right)(t)
$$

$$
= A \left( \pi(g)e_i^\pi \right)(t).
$$

That is, $A$ intertwines the irreducible representation $\pi$ and the subrepresentation $\lambda_{[\mathcal{H}_{[\pi]}]}$, where $\mathcal{H}_{[\pi], i} = \operatorname{span} \{ \phi_{ij}^\pi, \phi_{ij}^\pi, \ldots, \phi_{ij}^\pi \}$. Therefore Lemma 19 implies that $\pi \cong \lambda_{[\mathcal{H}_{[\pi]}]}$, and thus, $\lambda_{[\mathcal{H}_{[\pi]}]} \cong d_\pi \cdot \pi$. Since we’re assuming that $E_f$ is dense, we have

$$
\mathcal{H} \cong \bigoplus_{[\pi] \in \hat{G}} \mathcal{H}_{[\pi]},
$$
and we have proved (2).

We have now shown that \( \{ \frac{1}{d_{[\pi]}^2} \phi_{ij}^\pi \}_{[\pi] \in \mathcal{C}} \) forms an orthonormal basis for \( L^2(G) \). (Under the assumption that \( \mathcal{E}_f \) is dense in \( C(G) \).) Thus, if \( f \in L^2(G) \), we can write

\[
f = \sum c([\pi], i, j) \frac{1}{d_{[\pi]}^2} \phi_{ij}^\pi.
\]

Furthermore,

\[
\| f \|_2^2 = \sum_{[\pi] \in \mathcal{G}} \sum_{i,j=1}^{d_{[\pi]}} |c([\pi], i, j)|^2.
\]

Now (5) follows from the fact that

\[
c([\pi], i, j) = \int_G f(g) \frac{1}{d_{[\pi]}^2} \phi_{ij}^\pi(g) \, dg
\]

\[
= d_{[\pi]}^2 \int_G f(g) \langle \pi(g) e_j^\pi, e_i^\pi \rangle \, dg
\]

\[
= d_{[\pi]}^2 \langle \pi(f) e_j^\pi, e_i^\pi \rangle \, dg
\]

\[
= d_{[\pi]}^2 \langle \pi(f) \rangle_{ij}.
\]

Of course, (3) follows from (4) (otherwise, \( \mathcal{E} \) wouldn’t separate \( e \) and \( g \), so it only remains to prove that \( \mathcal{E}_f \) is dense in \( C(G) \). Towards this end, we need to recall some basic facts about so-called Hilbert-Schmidt operators. If \((X, M, \mu)\) is a measure space and if \( K \in L^2(X \times X, \mu \times \mu) \), then we can define a bounded linear operator \( T : L^2(X) \rightarrow L^2(X) \) by

\[
Tf(x) = \int_X K(x, y) f(y) \, dy.
\]

It is not hard to see that \( T \) is self-adjoint if \( K(x, y) = \overline{K(y, x)} \). An operator of this form is called a Hilbert-Schmidt operator and all such operators are self-adjoint compact operators\(^{(1)}\). In particular, each eigenspace

\[
\mathcal{H}_\alpha = \{ f \in L^2(X) : Tf = \alpha f \}
\]

\(^{(1)}\) In our case, we’ll only be interested in the case where \( X = G, \mu \) is normalized Haar measure, and \( K \) is continuous. Then the Stone-Weierstrass Theorem implies that there are functions \( \psi_i \in C(G) \) such that

\[
K(x, y) = \sum_{i \in I} \alpha_i \overline{\psi_i(x)} \psi_i(y)
\]
is finite dimensional and there is an orthonormal sequence \( \{ \phi_i \} \) of eigenvectors with eigenvalues \( \alpha_i \) so that every \( f \in L^2(X) \) can be written uniquely as

\[
f = \sum c_i \phi_i + \phi_0,
\]

where \( T\phi_0 = 0 \) and \( c_i = \langle f, \phi_i \rangle \).

Our interest in such operators is as follows. Let \( k \) be any element of \( C(G) \) which satisfies \( k(g) = \overline{k(g^{-1})} \). Therefore

\[
f * k(g) = \int_G f(r) k(r^{-1}g) \, dr = \int_G K(g, r) f(r) \, dr,
\]

where \( K(g, r) = k(r^{-1}g) \), is a self-adjoint Hilbert-Schmidt operator. Let \( C = \|K\|_\infty = \max_{x, y \in G} |k(y^{-1}x)| \). Notice that \( \|Tf\|_\infty \leq C\|f\|_1 \leq C\|f\|_2 \). A moment's reflection allows one to see that this implies that \( Tf \in C(G) \). (Of course, \( Tf \) is only defined almost everywhere, but I mean it agrees almost everywhere with a continuous function on \( G \). Since this function is uniquely determined, it makes sense to treat \( Tf \) as a continuous function. This is standard practice.) It follows that each eigenfunction of \( T \) is continuous.

**Lemma 21:** Let \( k \) be as above and let \( T \) be the Hilbert-Schmidt operator on \( L^2(G) \) defined by \( Tf = f * k \). Then for each \( \mu \in \mathbb{C} \setminus \{ 0 \} \)

\[
\mathcal{H}_\mu = \{ f \in L^2(G) : Tf = \mu f \} \subseteq \mathcal{E}_f.
\]

**Proof:** By the above remarks, \( \mathcal{H}_\mu \) is finite dimensional and consists of continuous functions. Suppose that \( f \in \mathcal{H}_\mu \). Then

\[
T(\lambda(g)f) = (\lambda(g)f) * k = \lambda(g)(f * k) = \lambda(g)(Tf) = \mu(\lambda(g)f).
\]

That is, \( \mathcal{H}_\mu \) is invariant for \( \lambda \). Let \( \{ f_1, \ldots, f_r \} \) be an orthonormal basis for \( \mathcal{H}_\mu \). Define continuous functions \( \psi_{ki} \) by

\[
\psi_{ki}(g) = \langle \lambda(g)f_i, f_k \rangle.
\]

Since \( \lambda(g)f_i \in \mathcal{H}_\mu \),

\[
f_i(g^{-1}t) = \sum_{k=1}^r \psi_{ki}(g) f_k(t).
\]

uniformly. Notice that for each finite subset \( F \subset I \) the operator \( T_F \) corresponding to

\[
K_F(x, y) = \sum_{i \in F} \alpha_i \overline{\psi_i(x)} \psi_i(y)
\]

is a finite rank operator. Since the convergence in (*) is uniform, it follows that \( T_F \to T \) in the operator norm; hence \( T \) is compact. (The remaining assertions in the paragraph follow from the Spectral Theorem.)
(A priori, Equation (2) is an equality in $L^2(G)$, and so would yeild pointwise equality only almost everywhere. But since both sides are continuous, the equality must hold everywhere.) Now define $\pi(g)$ to be the operator on $\mathcal{H}_\mu$ whose $r \times r$ matrix with respect to the basis $\{f_1, \ldots, f_r\}$ is $(\psi_{ij})$. Since we have $\psi_{ij}(g) = \overline{\psi_{ji}(g^{-1})}$, it follows from Equation (1) that $\pi(g)^* = \pi(g^{-1})$. Similarly,

$$\psi_{ij}(gt) = \langle \lambda(t)f_j, \lambda(g^{-1})f_i \rangle = \sum_{k=1}^r \langle f_k, \lambda(g^{-1})f_i \rangle \langle \lambda(t)f_j, f_k \rangle = \sum_{k=1}^r \psi_{ik}(g)\psi_{kj}(t).$$

Therefore $\pi(gt) = \pi(g)\pi(t)$. It follows that $\pi$ is a finite dimensional (unitary) representation of $G$. Using Equation (2), we see that

$$f_i(g) = \sum_{k=1}^r \psi_{ki}(g^{-1})f_k(e) = \sum_{k=1}^r f_k(e)\phi_i(g),$$

where $\phi_{ki}(g) = \psi_{ki}(g^{-1}) = \langle f_i, \lambda(g)f_k \rangle$. We have shown that each $f_i$, and hence $\mathcal{H}_\mu$, is in the span of the matrix coefficients of finite dimensional representations of $G$. Since every finite dimensional representation is the direct sum of irreducible (finite dimensional) representations, we have $\mathcal{H}_\mu \subseteq \mathcal{E}_f$ as desired. \hfill \Box

**Lemma 22:** Let $k$ and $T$ be as above. If $f \in L^2(G)$, then $Tf \in \overline{\mathcal{E}_f}$.

**Proof:** Let $\{\phi_i\}$ be a complete orthonormal set of eigenvectors for the eigenspaces with non-zero eigenvalues of $T$. By the spectral theorem, we can write

$$f = \sum_k c_k \phi_k + \phi_0,$$

where $T\phi_0 = 0$ and $\|f\|^2 \leq \sum_k |c_k|^2$. Let $T\phi_k = \alpha_k \phi_k$. Given $\varepsilon > 0$, there is an $N$ so that $\|\sum_{k>N} c_k \phi_k\| < \varepsilon$. In particular, $\|T\left(\sum_{k>N} c_k \phi_k\right)\| \leq C\varepsilon$. But

$$T\left(\sum_{k=1}^N c_k \phi_k\right) = \sum_{k=1}^N \alpha_k c_k \phi_k \in \mathcal{E}_f$$

11
by Lemma 21. This suffices as

$$
\|T f - \sum_{k=1}^{N} \alpha_k e_k \phi_k\|_\infty = \|T \left( \sum_{k>N} c_k \phi_k \right)\|_\infty \leq C \epsilon.
$$

The Peter-Weyl theorem now follows as $C(G)$ always contains a self-adjoint approximate identity. Specifically, we have the following.

**Proposition 23:** If $G$ is a compact group, then there is a net $\{ k_\alpha \}$ in $C(G)$ which satisfies $k_\alpha^* = k_\alpha$, and such that both $\{ k_\alpha * f \}$ and $\{ f * k_\alpha \}$ converge uniformly to $f$ for each $f \in C(G)$.

**Proof:** For each neighborhood $U$ of $e \in G$, let $k_U$ be a continuous non-negative function which satisfies $k_U(e) = 1$, $\int_G k_U(g) \, dg = 1$, $k_U^* = k_U$, and $\text{supp } k \subseteq U$. Then $\{ k_U \}_U$ is a net in $C(G)$ directed by reverse inclusion: i.e., $U \supseteq V$ if and only if $U \subseteq V$.

Fix $f \in C(G)$. Since $G$ is compact, given $\epsilon > 0$, there is a neighborhood $W$ of $e \in G$ such that $|f(t^{-1}g) - f(g)| < \epsilon$ for all $g \in G$ and $t \in W$. Therefore if $U \supseteq W$, then

$$
|k_U * f(g) - f(g)| = \left| \int_G k_U(t) f(t^{-1}g) \, dt - f(g) \right|
\leq \int_G k_U(t) \left| f(t^{-1}g) - f(g) \right| \, dt
\leq \epsilon \int_G k_U(t) \, dt = \epsilon.
$$

It follows that $k_U * f \to f$ uniformly, as claimed. On the other hand, $(k_U * f)^* = f^* * k_U$ implies that $f * k_U \to f$ uniformly as well. This proves the lemma.

Let $T_U$ be the Hilbert-Schmidt operator corresponding to $k_U$. Thus, $T_U f$ converges to $f$ uniformly for each $f \in C(G)$. Thus, $f \in \mathcal{E}_f$ by Lemma 22, and Theorem 17 is proved.
References