Arnold-type invariants
of curves and wave fronts on surfaces

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ABSTRACT


This thesis is devoted to the study of invariants of generic curves and wave fronts on surfaces. The invariants $J^\pm$ and $S^t$ were axiomatically defined by Arnold as numerical characteristics of generic curves (immersions of the circle) on $\mathbb{R}^2$. Recently he introduced $J^\pm$ in the case of generic planar wave fronts. The generalization of $S^t$ to this case was independently obtained by F. Alcardi and M. Polyak.

In the first two chapters of this thesis I construct generalizations of the three Arnold's invariants to the case of generic curves and wave fronts on an arbitrary surface (not necessarily $\mathbb{R}^2$). I explicitly describe all the invariants satisfying axioms, which naturally generalize the axioms used by Arnold.

To prove existence of these invariants I use certain properties of the fundamental group of the space of curves on a surface. All the homotopy groups of this space are calculated in the third chapter of the thesis.

Key words: Immersion of the circle, generic immersion, curves on surfaces, Legendrian immersion, Legendrian knot, wave fronts on surfaces, Whitney index, Maslov index, regular homotopy, perestroikas of plane curves and fronts, Arnold's basic invariants of plane curves and fronts, finite order invariants, Vassiliev invariants, homotopy groups of the space of curves on a surface.

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Contents

Introduction 1

I. Invariants of Curves on Surfaces 3

1. Arnold's Invariants 3
   1.1. Basic facts and definitions.
   1.2. Invariants St, J^+ and J^-.

2. Strangeness-type Invariant of Curves on Surfaces 5
   2.1. Natural decomposition of a triple point stratum.
   2.2. Axiomatic description of St.
   2.3. Singularity theory interpretation of St for orientable F.

3. J^+ -type Invariant of Curves on Surfaces 9
   3.1. Natural decomposition of a direct selftangency point stratum.
   3.2. Axiomatic description of J^+.
   3.3. Singularity theory interpretation of J^+ for orientable F.

4. J^- -type Invariant of Curves on Surfaces 11
   4.1. Natural decomposition of an inverse selftangency point stratum.
   4.2. Axiomatic description of J^-.
   4.3. Singularity theory interpretation of J^- for orientable F.

5. Proofs 14
   5.1. Proof of Theorem 2.2.C.
   5.2. Proof of Lemma 5.1.A.
   5.3. Proof of Lemma 5.1.B.
   5.4. Proof of Theorem 2.3.B.
   5.5. Proof of Theorem 3.2.B.
   5.6. Proof of Theorem 3.3.B.

II. Invariants of Wave Fronts on Surfaces 25

6. Invariants of Planar Fronts 25
   6.1. Basic facts and definitions.
   6.2. Invariants J^+, J^- and St'.
Introduction

Consider the space $\mathcal{F}$ of all curves (immersions of an oriented circle) on a surface $F$. We call a curve generic, if its only multiple points are double points of transversal selfintersection. Nongeneric curves form a discriminant hypersurface in $\mathcal{F}$. There are three main strata of the discriminant. They are formed by curves with a triple point; curves with a selftangency point, at which the velocity vectors of the two branches are pointing to the same direction (direct selftangency); and curves with a selftangency point, at which the velocity vectors of the two branches are pointing to the opposite directions (inverse selftangency). The union of these strata is dense in the discriminant. In [3] V. Arnold associated a sign to a generic crossing of each of these strata. He also introduced $St$, $J^+$ and $J^-$ invariants of generic curves on $\mathbb{R}^2$, which change by a constant under a positive crossing of a triple point, direct selftangency and inverse selftangency strata, respectively, and do not change under a crossing of the other two strata. These invariants estimate the number of crossings of each part of the discriminant, which are necessary to transform one generic curve on $\mathbb{R}^2$ to another.

In Chapter I of the thesis I construct generalizations\(^2\) of these invariants to the case of generic curves on any surface $F$ (not necessarily $\mathbb{R}^2$). The fact, that for most surfaces the fundamental group is nontrivial, allows us to subdivide each of the three strata of the discriminant into pieces. We show that this subdivision is natural from the point of view of the singularity theory. We take an integer valued function $\psi$ on the set of pieces obtained from one stratum, and try to construct an invariant which jumps by $\psi(P)$ under a positive crossing of $P$ and does not change under crossings of the other two strata. In an obvious sense $\psi$ is a derivative of such an invariant and the invariant is an integral of $\psi$. We introduce a condition on $\psi$ which is necessary and sufficient for existence of such an invariant. Any integrable, in the sense above, function $\psi$ defines this kind of an invariant up to an additive constant.

If the surface $F$ is orientable, then the condition which corresponds to the

\(^1\) Chapter I of this thesis is based on a paper Arnold-Type Invariants of Curves on Surfaces, submitted to the Journal of Knot Theory and its Ramifications

\(^2\) When the work described in Chapter I and Chapter III of the thesis was complete and the main results of it were published as preprints of Uppsala University [12] and [13], I received a preprint of A. Inshakov [9] containing similar results obtained by him independently.
generalizations of $J^+$ and $J^-$ is automatically satisfied and such an invariant exists for any function $\psi$. For the generalization of St the condition is not trivial. We reduce it to a simple condition on $\psi$ which is sufficient for existence of such an invariant. All these conditions are satisfied in the case of orientation reversing curves.

To prove the existence of these invariants I use certain properties of the fundamental group of the space of curves on a surface. All the homotopy groups of this space are calculated\(^3\)in Chapter III of the thesis.

The Arnold's invariants were generalized to the case of generic co-oriented oriented wave fronts on the plane. A wave front on $\mathbb{R}^2$ is the projection of a Legendrian curve in $ST\mathbb{R}^2$ (the spherical tangent bundle of $\mathbb{R}^2$) equipped with the natural coorientation and orientation. Arnold [5] constructed $J^\pm$ in the case of generic cooriented oriented planar fronts. A generalization of St was independently obtained by F. Aicardi [1] and M. Polyak [10]. Aicardi denoted this invariant by $Sp$ and Polyak by $St'$. The normalizations they used for this invariant are different, namely $Sp = 4 St'$. (In this paper I use Polyak's definition for the invariant.)

An approach, similar to the one I used to generalize Arnold's invariants of planar curves to the case of curves on an arbitrary surface, allowed me to generalize these invariants of wave fronts on $\mathbb{R}^2$ to the case of fronts on any surface. This is done in Chapter II of the thesis. There I present necessary and sufficient conditions for integrability, in the above sense, of functions on the parts of the strata of the discriminant. Similar to the case of curves the condition which corresponds to the generalizations of $J^+$ and $J^-$ is automatically satisfied, provided that the surface is orientable. In this case such invariants exists for any function. For the generalization of St the condition is not trivial. We reduce it to a simple condition on the function, which is sufficient for the existence of such an invariant.

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\(^3\)The results of the well known paper by S. Smale [11], where he calculated homotopy groups of immersed closed curves with the fixed initial point and the velocity vector at it, are not sufficient for this purpose.

\(^4\)Chapter III of this thesis is based on a paper Homotopy groups of the space of curves on a surface, accepted to Mathematica Scandinavica.
I. Invariants of Curves on Surfaces

1. Arnold’s Invariants

1.1. Basic facts and definitions. A curve is a smooth immersion of (an oriented circle) $S^1$ into a (smooth) surface $F$.

A generic immersion has only ordinary double points of transversal selfintersection. All nongeneric immersions form in the space of all immersions a discriminant hypersurface, or for short, the discriminant.

A selftangency point of (an oriented) curve is called a point of a direct selftangency, if the velocity vectors at this point have the same direction; otherwise it is called a point of an inverse selftangency.

A coorientation of a smooth hypersurface in a functional space is a local choice of one of the two parts, separated by this hypersurface, in a neighborhood of any of its points. This part is called positive.

The coorientation of the smooth part of a singular hypersurface is called consistent, if the following consistency condition holds in a neighborhood of any singular point of any stratum of codimension one on the hypersurface (of codimension two in the ambient functional space):

The intersection index of any generic small oriented closed curve with a hypersurface (defined as a difference between the numbers of positive and negative intersections) should vanish.

A hypersurface is called cooriented, if a consistent coorientation of its smooth part is chosen, and coorientable, if such a coorientation exists.

There are three parts of the discriminant hypersurface formed by the immersions having triple points, having direct selftangencies, and having inverse selftangencies, respectively.

1.1.A. LEMMA (ARNO LD [3]). Each of these three parts of the discriminant hypersurface is coorientable.

Consider a transversal crossing of a triple point stratum of the discriminant. A vanishing triangle is the triangle formed by the three branches of the curve corresponding to a subcritical or to a supercritical value of the parameter near the triple point of a critical curve.
The sign of a vanishing triangle is defined by the following construction. The orientation of the immersed circle defines the cyclic order on the sides of the vanishing triangle (it is the order of the visits of the triple point by the three branches). Hence, the sides of the triangle acquire orientations induced by the ordering. But each side has also its own orientation, which may coincide, or not, with the orientations defined by the ordering.

For each vanishing triangle we define a quantity \( q \in \{0, 1, 2, 3\} \) to be the number of sides of the vanishing triangle equally oriented by the ordering and their direction. The sign of the vanishing triangle is \((-1)^q\).

### 1.1.B. Definition of Sign of a Stratum Crossing.

A transversal crossing of a selftangency stratum of the discriminant is **positive**, if the number of double points grows (by two).

A transversal crossing of a triple point stratum of the discriminant is **positive**, if the new-born vanishing triangle is positive.

### 1.2. Invariants \( St, J^+ \) and \( J^- \).

The index of an immersion of an oriented circle into an oriented plane is the number of turns of the velocity vector. (The degree of the mapping sending a point of the circle to the direction of the derivative of the immersion at the point.) The Whitney's Theorem [16] says that the connected components of the space of oriented planar curves are counted by the indices of the curves.

Consider one of this components, that is, the space of immersions of a fixed index.

#### 1.2.A. Theorem (Arnold [3]).

There exists a unique (up to an additive constant) invariant of generic curves of a fixed index, whose value remains unchanged under a crossing of a selftangency stratum of the discriminant, but increases by one under a positive crossing of a triple point stratum of the discriminant.

This invariant is denoted by \( St \) (from Strangeness), when normalized by the following conditions:

\[
St(K_0) = 0, \quad St(K_{i+1}) = i \quad (i = 0, 1, \ldots),
\]

(1.1)

where \( K_0 \) is the figure eight curve and \( K_{i+1} \) is the simplest curve with \( i \) double points (see Figure 1). The curve \( K_j \) has index \( \pm j \), depending on the orientation.

![Figure 1](image-url)
1.2.B. THEOREM (ARNOLD [3]). There exists a unique (up to an additive constant) invariant of generic curves of a fixed index, whose value remains unchanged under a crossing of an inverse selftangency or of a triple point strata of the discriminant, but increases by two under a positive crossing of a direct selftangency stratum of the discriminant.

This invariant is denoted by \( J^+ \), when normalized by the following conditions:

\[
J^+(K_0) = 0, \quad J^+(K_{i+1}) = -2i \quad (i = 0, 1, \ldots),
\]

(1.2)

where \( K_0 \) and \( K_{i+1} \) are the curves shown in Figure 1.

1.2.C. THEOREM (ARNOLD [3]). There exists a unique (up to an additive constant) invariant of generic curves of a fixed index, whose value remains unchanged under a crossing of a direct selftangency or of a triple point strata of the discriminant, but decreases by two under a positive crossing of an inverse selftangency stratum of the discriminant.

This invariant is denoted by \( J^- \), when normalized by the following conditions:

\[
J^-(K_0) = -1, \quad J^-(K_{i+1}) = -3i \quad (i = 0, 1, \ldots),
\]

(1.3)

where \( K_0 \) and \( K_{i+1} \) are the curves shown in Figure 1.

2. Strangeness-type Invariant of Curves on Surfaces

2.1. Natural decomposition of a triple point stratum.

2.1.A. DEFINITION. Let \( F \) be a surface. We say that a curve \( \xi \subset F \) with a triple point \( q \) is a generic curve with a triple point, if its only nongeneric singularity is this triple point, at which every two branches are transverse to each other.

2.1.B. Let \( F \) be an oriented surface. Let \( B_3 \) be a bouquet of three oriented circles, with a fixed cyclic order on them, and \( b \) be the base point of \( B_3 \). Let \( s : S^1 \to F \) be a generic curve with a triple point \( q \). There exists an associated with \( s \) mapping \( \phi : B_3 \to F \), which satisfies the following conditions:

a) \( \phi(B_3) = s(S^1) \).
b) \( \phi(b) = q \).
c) \( \phi \) is injective on the preimage of the complement of the multiple points of the curve \( s \).
d) \( \phi \) is orientation preserving.
e) The cyclic order, induced on the circles of \( B_3 \) by traversing \( s(S^1) \) according to the orientation, coincides with the fixed one.

Note, that the free homotopy class of the mapping of \( B_3 \) to \( F \) realized by \( \phi \) is well defined, modulo an automorphism of \( B_3 \) which preserves the orientation and the cyclic order on the circles.
2.1.C. DEFINITION of T-EQUIVALENCE. Let \( s_1 \) and \( s_2 \) be generic curves with a triple point (see 2.1.A). We say, that these curves are T-equi


valent, if there exist associated with them mappings of \( B_3 \) which are free homotopic. A triple point stratum is naturally decomposed into parts corresponding to different T-


equivalence classes.

We denote by \([s]\) the T-equivalence class corresponding to \( s \), a generic curve with a triple point. We denote by \( \mathcal{T} \) the set of all the T-equivalence classes.

2.2. Axiomatic description of \( \mathcal{S}_T \).

2.2.A. Sliding of a kink. Let \( \xi \subset F \) be a generic curve, and \( \mathcal{F} \) be the space of all the curves on \( F \). Consider the loop \( \gamma_1 \) in \( \mathcal{F} \) constructed below.

Deform \( \xi \) along a generic path \( t \) in \( \mathcal{F} \) to get two opposite kinks, as it shown in Figure 2. Make the first kink very small and move it along the curve (in such a way that at each moment of time points of \( \xi \) located outside of a small neighborhood of the kink do not move) till it comes back. (See Figure 3.) Finally deform \( \xi \) to its original shape along \( t^{-1} \).

Note, that if \( \xi \) represents an orientation reversing loop on \( F \), then the kink slides two times along \( \xi \) before it returns to its original position.

![Figure 2](image)

![Figure 3](image)
2.2.B. Let $\xi$ be a generic curve and $\gamma_1 \subset F$ be the loop corresponding to the sliding of a kink along $\xi$ (see 2.2.A). We denote by $I^k$ the set of moments, when $\gamma_1$ crosses a triple point stratum. We denote by $\{t^k_i\}_{i \in I^k}$ the $T$-equivalence classes corresponding to the parts of the stratum, where the crossings occur, and by $\{\sigma^k_i\}_{i \in I^k}$ the signs of the crossings.

2.2.C. Theorem. Let $F$ be a surface (not necessarily compact or orientable), $T$ be the set of all the $T$-equivalence classes and $C$ be a connected component of $F$. Let $\psi : T \to \mathbb{Z}$ be a function.

Then the following two statements I and II are equivalent.

I: There exists an invariant $\tilde{\psi}$ of generic immersions in $C$, such that:

a) Its value does not change under a crossing of selftangency strata of the discriminant.

b) Its value increases by $\psi([s])$ under a positive crossing of the part of a triple point stratum, which corresponds to a $T$-equivalence class $[s]$.

II: For some generic immersion $\xi \in C$

\[ \sum_{i \in I^k} \sigma^k_i \psi(t^k_i) = 0 \] (2.1)

(see 2.2.B).

For the Proof of Theorem 2.2.C see Section 5.1.

Remark. If for a given function $\psi : T \to \mathbb{Z}$ there exists an invariant $\tilde{\psi}$ satisfying conditions a) and b) of statement I of Theorem 2.2.C, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 2.2.C.)

Theorem 2.2.C implies, that if statement II of Theorem 2.2.C is true for a generic immersion $\xi \in C$, then it is true for any generic immersion $\xi' \in C$.

2.2.D. Cases, when the condition on $\psi$ is trivial. Let $\xi$ be a generic curve on $F$. Clearly, all the crossings of a triple point stratum, which occur under the sliding of a kink along $\xi$ happen, when the kink passes through a double point of $\xi$.

If $F$ is orientable, then the kink passes two times through each double point. A straightforward check shows that the signs of the corresponding triple point stratum crossings are opposite. The mappings of $B_3$, associated with these crossings, are different by an orientation preserving automorphism of $B_3$, which does not preserve the cyclic order on the circles. For both crossings the restriction of an associated mapping to one of the circles of $B_3$ represents a homotopically trivial loop. Thus, if $F$ is orientable then statement II (and hence statement I) of Theorem 2.2.C is true for any function $\psi$, provided that it takes the same value on any two $T$-equivalence classes, for which there exist $\phi_1, \phi_2$ mappings of $B_3$ representing them, such that:

a) The restriction of $\phi_i$ ($i \in \{1,2\}$) to one of the circles of $B_3$ represents a homotopically trivial loop on $F$.

b) There exists $\alpha$, an orientaion preserving automorphism of $B_3$ (not preserving the cyclic order on the circles) such that $\phi_1 = \phi_2 \circ \alpha$. 
If $\xi$ represents an orientation reversing loop on $F$, then the kink has to slide two times along $\xi$ before it comes to its original position. Thus, it passes four times through each double point of $\xi$. One can show, that the corresponding crossings of a triple point stratum can be subdivided into two pairs, such that the $T$-equivalence classes corresponding to the crossings inside each pair are equal and the signs of the two crossings in each pair are opposite. Thus, if $\xi$ represents an orientation reversing loop on $F$, then statement II (and hence statement I) of Theorem 2.2.C is true for any function $\psi : T \to \mathbb{Z}$. (Another way of proving this is based on the fact that for such $\xi$ the loop $\gamma_1 = 1 \in \pi_1(F, \xi)$, see Section 5.3.J.)

2.2.E. **Connection with the standard $\overline{St}$-invariant.** Since $\pi_1(\mathbb{R}^2) = 0$ there is just one $T$-equivalence class of singular immersions to $\mathbb{R}^2$. Thus, the construction of $\overline{St}$ does not give anything new in the classical case of planar curves.

2.3. **Singularity theory interpretation of $\overline{St}$ for orientable $F$.**

2.3.A. **Definition of $\overline{T}$-equivalence.** Let $S^1(3)$ be the configuration space of unordered triples of points on $S^1$. Consider a space $S^1(3) \times F$. Let $\mathcal{M}$ be the subspace of $S^1(3) \times F$ consisting of $t \times f \in S^1(3) \times F$, such that $f$ maps the three points from $t$ to one point on $F$. (This is a sort of singularity resolution for strata involving points of multiplicity greater than two.)

We say that $m_1, m_2 \in \mathcal{M}$ are $\overline{T}$-equivalent, if they belong to the same path connected component of $\mathcal{M}$. For $s$, a generic curve with a triple point (see 2.1.A), there is a unique $\overline{T}$-equivalence class associated to it. We denote this class by $[s]$.

Thus, the $\overline{T}$-equivalence relation induces a decomposition of a triple point stratum of the discriminant hypersurface.

Let $\overline{T}$ be the set of all the $\overline{T}$-equivalence classes. There is a natural mapping $\phi : \overline{T} \to \mathcal{T}$. It maps $\overline{t} \in \overline{T}$ to such $t \in \mathcal{T}$, that there exists $s$, a generic curve with a triple point, for which $[s] = t$ and $[\overline{s}] = \overline{t}$.

Let $F$ be an orientable surface. Let $C$ be a connected component of $F$ and $\overline{T}_C \subseteq \overline{T}$ be the set of all the $\overline{T}$-equivalence classes, corresponding to some generic curves (from $C$) with a triple point.

2.3.B. **Theorem.** The mapping $\phi |_{\overline{T}_C}$ is injective.

For the Proof of Theorem 2.3.B see Section 5.4.

2.3.C. **Interpretation of $\overline{St}$.** Let $F$ be an orientable surface. Let $C$ be a connected component of $F$. Let $\overline{St}$ be an invariant of generic immersions in $C$ such that:

a) It does not change under the crossing of selftangency strata of the discriminant.

b) Under the crossing of a part of a triple point stratum of the discriminant it jumps by a constant dependent only on the $\overline{T}$-equivalence class corresponding to this part of the discriminant.

Theorem 2.3.B implies that this $\overline{St}$ invariant is an $\overline{St}$ invariant for some choice of the function $\psi : T \to \mathbb{Z}$. 
3. \( J^+ \)-type Invariant of Curves on Surfaces

3.1. Natural decomposition of a direct selftangency point stratum.

3.1.A. Definition. Let \( F \) be a surface. We say that a curve \( \xi \subset F \) with a direct selftangency point \( q \) is a generic curve with a direct selftangency point, if its only nongeneric singularity is this point.

3.1.B. Let \( F \) be a surface. Let \( B_2 \) be a bouquet of two oriented circles, and \( b \) be its base point. Let \( s : S^1 \to F \) be a generic curve with a direct selftangency point \( q \). It can be lifted to the mapping \( \bar{s} \) from the oriented circle to \( STF \) (the spherical tangent bundle of \( F \)), which sends a point \( p \in S^1 \) to the point in \( STF \), to which points the velocity vector of \( s \) at \( p \). (Note, that \( q \) lifts to a double point \( \bar{q} \) of \( \bar{s} \).) There exists an associated with \( s \) mapping \( \phi : B_2 \to STF \), which satisfies the following conditions:

a) \( \phi \) is injective and \( \phi(S^1) = \bar{s}(S^1) \).

b) \( \phi(b) = \bar{q} \).

c) \( \phi \) is orientation preserving.

Note, that the free homotopy class of a mapping of \( B_2 \) to \( STF \) realized by \( \phi \) is well defined, modulo the orientation preserving automorphism of \( B_2 \) which interchanges the circles.

3.1.C. Definition of \( T^+ \)-equivalence. Let \( s_1 \) and \( s_2 \) be generic curves with a point of direct selftangency (see 3.1.A). We say that these curves are \( T^+ \)-equivalent, if there exist associated with the two of them mappings of \( B_2 \), which are free homotopic. A direct selftangency point stratum is naturally decomposed into parts corresponding to different \( T^+ \)-equivalence classes.

We denote by \([s^+]\) the \( T^+ \)-equivalence class corresponding to \( s \), a generic curve with a point of direct selftangency. We denote by \( T^+ \) the set of all the \( T^+ \)-equivalence classes.

3.2. Axiomatic description of \( J^+ \).

3.2.A. Let \( \xi \) be a generic curve and \( \gamma_1 \subset F \) be the loop corresponding to the sliding of a kink along \( \xi \) (see 2.2.A). We denote by \( J^\xi \) the set of moments, when \( \gamma_1 \) crosses a direct selftangency point stratum. We denote by \( \{\sigma_j^\eta\}_{\eta\in H} \) the \( T^+ \)-equivalence classes corresponding to the parts of the stratum, where the crossings occur, and by \( \{\sigma_j^\eta\}_{\eta\in H} \) the signs of the crossings.

3.2.B. Theorem. Let \( F \) be a surface (not necessarily compact or orientable), \( T^+ \) be the set of all the \( T^+ \)-equivalence classes and \( C \) be a connected component of \( F \). Let \( \psi : T^+ \to \mathbb{Z} \) be a function. Then the following two statements I and II are equivalent.

I: There exists an invariant \( J^+ \) of generic immersions in \( C \), such that:

a) Its value does not change under a crossing of an inverse selftangency and of a triple point strata of the discriminant.

b) Its value increases by \( 2\psi([s^+]^+) \) under a positive crossing of the part of a direct selftangency stratum, which corresponds to a \( T^+ \)-equivalence class \([s^+]^+\).
II: For some generic immersion $\xi \in C$

$$\sum_{\xi \in J^+} \sigma^T_\xi \psi(t_\xi) = 0$$  (3.1)

(see 3.2.A).

For the Proof of Theorem 3.2.B see Section 5.5.

Remark. If for a given function $\psi : T \to \mathbb{Z}$ there exists an invariant $J^+$ satisfying conditions a) and b) of statement I of Theorem 3.2.B, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 3.2.B.)

Theorem 3.2.B implies, that if statement II of Theorem 3.2.B is true for a generic immersion $\xi \in C$, then it is true for any generic immersion $\xi' \in C$.

3.2.C. Cases, when the condition on $\psi$ is trivial. Let $\xi$ be a generic curve on $F$. Clearly, all the crossings of a direct selftangency point stratum, which occur under the sliding of a kink along $\xi$, happen when the kink passes through a double point of $\xi$.

If $F$ is orientable, then the kink passes twice through each double point of $\xi$. A straightforward check shows that the signs of the corresponding direct selftangency stratum crossings are different, and that the $T^+$-equivalence classes corresponding to them are equal. Thus, if $F$ is orientable, then statement II (and hence statement I) of Theorem 3.2.B is true for any function $\psi : T^+ \to \mathbb{Z}$.

If $\xi$ represents an orientation reversing loop on $F$, then the kink slides two times along $\xi$, before it comes to its original position. Thus, it passes four times through each double point of $\xi$. One can show, that the corresponding four crossings of a direct selftangency stratum can be subdivided into two pairs, such that the $T^+$-equivalence classes corresponding to the crossings inside the same pair are equal and the signs of the two crossings in each pair are opposite. Thus, if $\xi$ represents an orientation reversing loop on $F$, then statement II (and hence statement I) of Theorem 3.2.B is true for any function $\psi : T^+ \to \mathbb{Z}$. (Another way of proving this is based on the fact, that for such $\xi$ the loop $\gamma_1 = 1 \in \pi_1(F, \xi)$, see Section 5.3.J.)

3.2.D. Connection with the standard $J^+$-invariant. Since $\pi_1(ST\mathbb{R}^2)$ is isomorphic to $\mathbb{Z}$, there are countably many $T^+$-equivalence classes of singular immersions to $\mathbb{R}^2$, which can be obtained from a curve of the fixed index. (Note, that the index of an immersed curve $C$ defines the connected component of the space of all curves on $\mathbb{R}^2$, where $C$ belongs to.) Thus, the construction of $J^+$ gives rise to a splitting of the standard $J^+$ invariant of V. Arnold. This is the splitting introduced by V. Arnold [4] in the case of planar curves of index zero and generalized to the case of arbitrary planar curves by P. Aicardi [2].
3.3. Singularity theory interpretation of $\mathcal{F}^+$ for orientable $F$.

3.3.A. Definition of $\mathcal{T}^+$-equivalence. Put $S^1(2)$ to be the configuration space of unordered pairs of points on $S^1$. Consider a space $S^1(2) \times \mathcal{F}$. Let $\mathcal{M}^+$ be the subspace of $S^1(2) \times \mathcal{F}$ consisting of $t \times f \in S^1(2) \times \mathcal{F}$, such that $f$ maps the two points from $t$ to one point on $F$ and the velocity vectors of $f$ at these two points have the same direction. (This is a sort of singularity resolution for the strata involving points of direct selftangency.)

We say that $m_1^+$ and $m_2^+$ from $\mathcal{M}^+$ are $\mathcal{T}^+$-equivalent, if they belong to the same path connected component of $\mathcal{M}^+$.

Clearly, for $s$, a generic curve with a direct selftangency point (see 3.1.A), there is a unique $\mathcal{T}^+$-equivalence class associated with it. We denote this class by $[s^+]$. Thus, the $\mathcal{T}^+$-equivalence relation induces a decomposition of a direct selftangency point stratum of the discriminant hypersurface.

Let $\mathcal{T}^+$ be the set of all the $\mathcal{T}^+$-equivalence classes. There is a natural mapping $\phi: \mathcal{T}^+ \to \mathcal{T}^+$. It maps $t^+ \in \mathcal{T}^+$ to such $s^+ \in \mathcal{T}^+$, that there exists $s$ (a generic immersion with a direct selftangency point), for which $[s^+] = t^+$ and $[s^+] = t^+$.

Let $F$ be an orientable surface. Let $C$ be a connected component of $\mathcal{F}$ and $\mathcal{T}^+ \subset \mathcal{T}^+$ be the set of all the $\mathcal{T}^+$-equivalence classes corresponding to generic curves (from $C$) with a point of direct selftangency.

3.3.B. Theorem. The mapping $\phi|_{\mathcal{T}^+}$ is injective.

For the proof of Theorem 3.3.B see Section 5.6.

3.3.C. Interpretation of $\mathcal{J}^+$. Let $F$ be an orientable surface. Let $C$ be a connected component of $\mathcal{F}$. Let $\mathcal{J}^+$ be an invariant of generic immersions in $C$, such that:

a) It does not jump under a crossing of an inverse selftangency and of a triple point strata of the discriminant.

b) Under a crossing of a part of a direct selftangency point stratum of the discriminant it jumps by a constant, dependent only on the $\mathcal{T}^+$ equivalence class corresponding to this part of the discriminant.

Theorem 3.3.B implies that this $\mathcal{J}^+$ invariant is a $\mathcal{J}^+$ invariant for some choice of the function $\psi: \mathcal{T}^+ \to \mathbb{Z}$.

4. $\mathcal{J}^-$-type Invariant of Curves on Surfaces

4.1. Natural decomposition of an inverse selftangency point stratum.

4.1.A. Definition. Let $F$ be a surface. We say that a curve $\xi \subset F$ with an inverse selftangency point $q$ is a generic curve with an inverse selftangency point, if its only nongeneric singularity is this point.

4.1.B. Let $F$ be a surface. Let $B_2$ be a bouquet of two oriented circles, and $b$ be its base point. Let $s: S^2 \to F$ be an oriented generic curve with a point of an inverse selftangency $q$. It can be lifted to the mapping $\tilde{s}$ from the oriented circle to $PTF$ (the projectivized tangent bundle of $F$) which sends a point $p \in S^2$ to
the point in \( PT \), to which points the line tangent to \( s \) at \( s(p) \). (Note, that \( q \) lifts to a double point \( \bar{q} \) of \( \bar{s} \).) There exists an associated with \( s \) mapping \( \phi : B_2 \rightarrow PT \), which satisfies the following conditions:

- a) \( \phi \) is injective and \( \phi(B_2) = \bar{s}(S^1) \).
- b) \( \phi(b) = \bar{q} \).
- c) \( \phi \) is orientation preserving.

Note, that the free homotopy class of a mapping of \( B_2 \) to \( PT \) realized by \( \phi \) is well defined, modulo the orientation preserving automorphism of \( B_2 \) which interchanges the circles.

4.1.C. Definition of \( T^- \)-equivalence. Let \( s_1 \) and \( s_2 \) be generic curves with a point of an inverse self-tangency (see 4.1.A). We say, that these curves are \( T^- \)-equivalent, if there exist associated with the two of them mappings of \( B_2 \), which are free homotopic. An inverse self-tangency point stratum is naturally decomposed into parts corresponding to different \( T^- \)-equivalence classes.

We denote by \( [s^-] \) the \( T^- \)-equivalence class corresponding to \( s \), a generic curve with a point of an inverse self-tangency. We denote by \( T^- \) the set of all the \( T^- \)-equivalence classes.

4.2. Axiomatic description of \( T^- \).

4.2.A. Let \( \xi \) be a generic curve and \( \gamma_1 \subset F \) be the loop corresponding to the sliding of a kink along \( \xi \) (see 2.2.A). We denote by \( F \) the set of moments, when \( \gamma_1 \) crosses an inverse self-tangency point stratum. We denote by \( \{t_i^\xi\}_{i \in I} \) the \( T^- \)-equivalence classes corresponding to the parts of the stratum, where the crossings occur, and by \( \{\sigma_i^\xi\}_{i \in I} \) the signs of the crossings.

4.2.B. Theorem. Let \( F \) be a surface, \( T^- \) be the set of all the \( T^- \)-equivalence classes and \( \mathcal{C} \) be a connected component of \( F \). Let \( \psi : T^- \rightarrow \mathbb{Z} \) be a function. Then the following two statements I and II are equivalent:

I: There exists an invariant \( \bar{T}^- \) of generic immersions in \( C \), such that:
- a) Its value does not change under a crossing of a direct self-tangency and of a triple point strata of the discriminant.
- b) Its value increases by \( -2\psi([s^-]) \) under a positive crossing of the part of an inverse self-tangency stratum which corresponds to a \( T^- \)-equivalence class \([s^-]\).

II: For some generic immersion \( \xi \in C \)

\[
\sum_{i \in I} \sigma_i^\xi \psi(t_i^\xi) = 0
\]

(see 4.2.A).

The Proof of Theorem 4.2.B is a straightforward generalization of the Proof of Theorem 3.2.B.

Remark. If for a given function \( \psi : T^- \rightarrow \mathbb{Z} \) there exists an invariant \( \bar{T}^- \) satisfying conditions a) and b) of statement I of Theorem 4.2.B, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 4.2.B.)

Theorem 4.2.B implies, that if statement II of Theorem 4.2.B is true for a generic immersion \( \xi \in C \), then it is true for any generic immersion \( \xi' \in C \).
4.2.C. Cases, when the condition on $\psi$ is trivial. Similarly to section 3.2.C one can show, that statement II (and hence statement I) of Theorem 4.2.B is true for any function $\psi : \mathcal{T}^- \to \mathbb{Z}$, provided that $\mathcal{F}$ is orientable, or that $\mathcal{C}$ consists of curves, realizing orientation reversing loops on $\mathcal{F}$.

4.2.D. Connection with the standard $J^-$ invariant. Since $\pi_1(PT\mathbb{R}^2)$ is isomorphic to $\mathbb{Z}$, there are countably many $\mathcal{T}^-$-equivalence classes of singular immersions, which can be obtained from a curve of the fixed index. (Note, that the index of an immersed curve $\mathcal{C}$ defines the connected component of the space of all curves on $\mathbb{R}^2$, where $C$ belongs to.) Thus, the construction of $\overline{J^-}$ gives rise to a splitting of the standard $J^-$ invariant of V. Arnold. This splitting is analogous to the splitting of $J^+$ introduced by V. Arnold [4] in the case of planar curves of index zero and generalized to the case of arbitrary planar curves by F. Aicardi [2].

4.3. Singularity theory interpretation of $\overline{J^-}$ for orientable $\mathcal{F}$.

4.3.A. Definition of $\overline{\mathcal{T}^-}$-equivalence. Put $S^1(2)$ to be the configuration space of unordered pairs of points on $S^1$. Consider a space $S^1(2) \times \mathcal{F}$. Let $\mathcal{M}^-$ be the subspace of $S^1(2) \times \mathcal{F}$ consisting of $t \times f \in S^1(2) \times \mathcal{F}$, such that $f$ maps the two points from $t$ to one point on $\mathcal{F}$ and the velocity vectors of $f$ at these two points have opposite directions. (This is a sort of singularity resolution for the strata involving points of a reverse self-tangency.)

We say that $m_1^-$ and $m_2^-$ from $\mathcal{M}^-$ are $\overline{\mathcal{T}^-}$-equivalent, if they belong to the same path connected component of $\mathcal{M}^-$. Clearly, for $s$, a generic curve with an inverse self-tangency point (see 4.1.A), there is a unique $\overline{\mathcal{T}^-}$-equivalence class associated to it. We denote it by $[s^-]$. Thus, the $\overline{\mathcal{T}^-}$-equivalence relation induces a decomposition of an inverse self-tangency point stratum of the discriminant hypersurface.

Let $\overline{\mathcal{T}^-}$ be the set of all the $\overline{\mathcal{T}^-}$-equivalence classes. There is a natural mapping $\phi : \overline{\mathcal{T}^-} \to \mathcal{T}^-$. It maps $\overline{t^-} \in \overline{\mathcal{T}^-}$ to such $t^- \in \mathcal{T}^-$, that there exists $s$ (a generic curve with an inverse self-tangency point), for which $[s^-] = t^-$ and $[s^-] = t^-$. Let $\mathcal{F}$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$, and $\overline{\mathcal{T}^-}_\mathcal{C} \subset \overline{\mathcal{T}^-}$ be the set of all the $\overline{\mathcal{T}^-}$-equivalence classes corresponding to some generic curves (from $\mathcal{C}$) with a point of an inverse self-tangency.

4.3.B. Theorem. The mapping $\phi|_{\overline{\mathcal{T}^-}_\mathcal{C}}$ is injective.

The Proof of Theorem 4.3.B is a straightforward generalization of the proof of Theorem 5.6.

4.3.C. Interpretation of $\overline{J^-}$. Let $\mathcal{F}$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$. Let $\overline{J^-}$ be an invariant of generic immersions in $\mathcal{C}$, such that:

a) It does not jump under a crossing of a direct self-tangency and of a triple point strata of the discriminant.
b) Under a crossing of a part of an inverse selftangency point stratum of the discriminant it jumps by a constant, dependent only on the $T^\infty$-equivalence class corresponding to this part of the discriminant.

Theorem 4.3.B implies, that this $\tilde{J}^-$ invariant is a $\overline{J}^-$ invariant for some choice of the function $\psi : T^\infty \to \mathbb{Z}$.

5. Proofs

5.1. Proof of Theorem 2.2.C. Clearly, in order for $\overline{S}^\infty(\xi)$ to be well defined, the sum of the jumps of it under the sliding of a kink (see 2.2.A) has to be zero. But this sum is exactly the sum from the statement II. Thus, we proved that statement I implies statement II.

Let us prove now that statement II implies statement I.

Fix a connected component $C$ of the space of all immersions and a generic immersion $\xi \in C$. Fix any value of $\overline{S}^\infty(\xi) \in \mathbb{Z}$. Let $\xi' \in C$ be another generic immersion. Take a generic path $p$ in $C$, which connects $\xi$ with $\xi'$. When we go along this path we see a sequence of crossings of a selftangency and of a triple point strata of the discriminant. Let $I$ be the set of moments when we crossed a triple point stratum. Let $\{\sigma_i\}_{i \in I}$ be the signs of the corresponding new born vanishing triangles and $\{[s_i]_{i \in I}$ be the $T$-equivalence classes represented by the corresponding generic curves with a triple point. Put $\Delta_{\overline{S}^\infty}(p) = \sum_{i \in I} \sigma_i \psi([s_i])$ and $\overline{S}^\infty(\xi') = \overline{S}^\infty(\xi) + \Delta_{\overline{S}^\infty}(p)$. To prove the Theorem it is sufficient to show, that $\overline{S}^\infty(\xi')$ does not depend on the generic path $p$, we used to define it. The last statement follows from Lemma 5.1.A and Lemma 5.1.B. Thus, we proved Theorem 5.1 modulo these two lemmas. □

5.1.A. Lemma. Let $p$ be a generic path in $F$, which connects $\xi$ to itself. Then $\Delta_{\overline{S}^\infty}(p)$ depends only on the element of $\pi_1(F, \xi)$ represented by $p$.

5.1.B. Lemma. If statement II of Theorem 2.2.C is true, then for every element of $\pi_1(F, \xi)$ there exists a generic loop $q$ in $F$, representing this element, such that $\Delta_{\overline{S}^\infty}(q) = 0$.

5.2. Proof of Lemma 5.1.A. It is sufficient to show that if we go around any stratum of codimension two along a small generic loop $r$ (not necessarily starting at $\xi$), then $\Delta_{\overline{S}^\infty}(r) = 0$. The only strata of codimension two in the bifurcation diagram of which triple points are present are: a) two distinct triple points, b) triple point at which two branches are tangent (of order one) and c) quadruple point (at which every two branches are transverse).

If $r$ is a small loop which goes around a stratum of two distinct triple points, then in $\Delta_{\overline{S}^\infty}(r)$ we have each of the two $T$-equivalence classes twice, once with the plus sign of the newborn vanishing triangle, once with the minus. Hence $\Delta_{\overline{S}^\infty}(r) = 0$.

Let $r$ be a small loop which goes around a stratum of a triple point with two tangent branches. We can assume, that it corresponds to a loop on Figure 4 directed clockwise. (The colored triangles are the newborn vanishing triangles.) As we can see from Figure 4, there are just two terms in $\Delta_{\overline{S}^\infty}(r)$. It is clear, that the $T$-equivalence classes in them coincide. A direct check shows that the signs
of the two terms are opposite. (Note, that if they are not always opposite, then Arnold's St invariant is not well defined.)

![Figure 4](image)

Finally, let \( r \) be a small loop, which goes around a stratum of a quadruple point (at which every two branches are transverse). We can assume, that it corresponds to a loop in Figure 5 directed counter clockwise. There are eight terms in \( \Delta_{ST}(r) \). We split them into pairs I, II, III, IV, as it is shown in Figure 5. One can see, that the \( T \)-equivalence classes of the two curves in each pair are the same. For each branch the sign of the colored triangle is equal to the sign of the triangle, which died under the triple point stratum crossing shown on the next (in the counterclockwise direction) branch. The sign of the dying vanishing triangle is minus the sign of the newborn vanishing triangle. Finally, one can see that the signs of the colored triangles inside each pair are opposite. Thus, all these eight terms cancel out.

This finishes the Proof of Lemma 5.1.A. \( \square \)

5.3. Proof of Lemma 5.1.B.

5.3.A. Remark. In \( \mathbb{Z} \) there are no elements of finite order. Thus, if \( m \neq 0 \), then \( \Delta_{ST}(q) \neq 0 \Leftrightarrow m\Delta_{ST}(q) = \Delta_{ST}(q^m) \neq 0 \). Hence, to prove Lemma 5.1.B it is sufficient to show that \( \Delta_{ST}(q^m) = 0 \) for a certain power \( m \neq 0 \) of \( q \in \pi_1(F, \xi) \).

5.3.B. Proposition. Let \( F \) be a surface, \( STF \) be its spherical tangent bundle and \( p \in STF \) be a point. Let \( f \in \pi_1(STF, p) \) be the class of an oriented (in some way) fiber of the \( S^1 \)-fibration \( pr : STF \to F \).

If \( \alpha \in \pi_1(STF, p) \) is a loop projecting to an orientation preserving loop on \( F \), then

\[ \alpha f = f \alpha. \]  \hspace{1cm} (5.1)
If \( \alpha \in \pi_1(STF, p) \) is a loop projecting to an orientation reversing loop on \( F \), then
\[
\alpha f = f^{-1} \alpha.
\] (5.2)

The proof of this Proposition is straightforward.

5.3.C. Parametric h-principle. The parametric h-principle (see [7] page 16) implies that \( \mathcal{F} \) is weak homotopy equivalent to the space \( \Omega STF \) of free loops in \( STF \). The corresponding mapping \( h : \mathcal{F} \to \Omega STF \) sends an immersion \( \xi \in \mathcal{F} \) to a loop \( \tilde{\xi} \in \Omega STF \) by mapping a point \( y \in S^1 \) to a point in \( STF \), to which points the velocity vector of \( \xi \) at \( \xi(y) \).

Let \( \tilde{\xi} \) be the lifting of \( \xi \) to a loop in \( STF \). Fix a point \( a \) on \( S^1 \). Let \( q \) be a loop in \( \mathcal{F} \) starting at \( \xi \). Then \( q_h : S^1 \times S^1 \to STF \) (the lifting of \( q \) by \( h \)) restricted to \( a \times S^1 \) gives rise to the loop \( t_a(q) \subseteq STF \) (lifting of the trajectory of \( a \) under \( q \)).

Note, that if \( q \in \pi_1(\mathcal{F}, \xi) \) is the sliding of a kink (see 2.2.A) along a curve \( \xi \) representing an orientation preserving loop on \( F \), then the velocity vector of \( \xi \) at \( \xi(a) \) is rotated by \( 2\pi \) under this sliding. Thus, \( t_a(q) \in \pi_1(STF, \tilde{\xi}(a)) \) is equal to \( f \), the homotopy class of the fiber of the \( S^1 \)-fibration \( pr : STF \to F \).

One can check, that if \( q \in \pi_1(\mathcal{F}, \xi) \) is the sliding of a kink along a curve \( \xi \) representing an orientation reversing loop on \( F \), then \( t_a(q) = 1 \in \pi_1(STF, \tilde{\xi}(a)) \). (In this case the kink has to slide twice along \( \xi \), before it returns to its original position and the total angle of rotation of a velocity vector of \( \xi \) at \( a \) appears to be zero.)

5.3.D. Proposition. The group \( \pi_1(\Omega STF, \lambda) \) is isomorphic to \( Z(\lambda) \), the centralizer of an element \( \lambda \in \pi_1(STF, \lambda(a)) \).
5.3.E. Proof of Proposition 5.3.D. Let \( p : \Omega STF \to STF \) be the mapping, which sends \( \omega \in \Omega STF \) to \( \omega(a) \in STF \). (One can check, that this \( p \) is a Serre fibration, with the fiber of \( p \) isomorphic to the space of loops based at the corresponding point.)

A Proposition proved by V.L. Hansen [8] says that: if \( X \) is a topological space with \( \pi_1(X) = 0 \), then \( \pi_1(\Omega X, \omega) = Z(\omega) < \pi_1(X, \omega(a)) \). (Here \( \Omega X \) is the space of free loops in \( X \) and \( \omega \) is an element of \( \Omega X \).) One can check that \( \pi_2(STF) = 0 \) for any surface \( F \). Thus, we get that \( \pi_1(\Omega STF, \lambda) \) is isomorphic to \( Z(\lambda) < \pi_1(STF, \lambda(a)) \). From the proof of the Hansen’s Proposition it follows that the isomorphism is induced by \( p_\lambda \).

The following statement is an immediate consequence of Proposition 5.3.D and the \( h \)-principle (see Section 5.3.C).

5.3.F. Corollary. Let \( F \) be a surface and \( \xi \) be a curve on \( F \), then \( \pi_2(F, \xi) \) is isomorphic to \( Z(\xi) \), the centerizer of \( \xi \in \pi_1(STF, \xi(a)) \). The isomorphism is given by the mapping \( t_\alpha : \pi_1(F, \xi) \to Z(\xi) \), which sends \( q \in \pi_1(F, \xi) \) to \( t_\alpha(q) \). (See Section 5.3.C.)

5.3.G. Proposition. Let \( F \neq S^2, T^2 \) (torus), \( \mathbb{RP}^2, K \) (Klein bottle) be a surface (not necessarily compact or orientable) and \( G' \) be a nontrivial commutative subgroup of \( \pi_1(F) \). Then \( G' \) is infinite cyclic and there exists a unique maximal subgroup \( G < \pi_1(F) \), such that \( G' < G \) and that \( G \) is isomorphic to \( Z \).

5.3.H. Proof of Proposition 5.3.G. It is well known, that any closed \( F \), other than \( S^2, T^2, \mathbb{RP}^2, K \), admits a hyperbolic metric of a constant negative curvature. (It is induced from the universal covering of \( F \) by the hyperbolic plane \( H \).) The Theorem by A. Freiessmann (see [6] pp. 258-265) says, that if \( M \) is a compact Riemannian manifold with a negative curvature, then any nontrivial Abelian subgroup \( G' < \pi_1(M) \) is isomorphic to \( Z \). Thus, if \( F \neq S^2, T^2, \mathbb{RP}^2, K \) is closed, then any nontrivial commutative \( G' < \pi_1(F) \) is infinite cyclic.

The proof of the Freiessmann’s Theorem given in [6] is based on the fact, that if \( \alpha, \beta \in \pi_1(M) \) are nontrivial commuting elements, then there exists a geodesic in \( \tilde{M} \) (the universal covering of \( M \)) which is mapped to itself under the action of these elements considered as deck transformations on \( \tilde{M} \). Moreover, these transformations restricted to the geodesic act as translations. This implies, that if \( F \neq S^2, T^2, \mathbb{RP}^2, K \) is a closed surface, then there exists a unique maximal infinite cyclic \( G < \pi_1(F) \), such that \( G' < G \). This gives the proof of Proposition 5.3.G for closed \( F \).

If \( F \) is not closed then the statement of the Proposition is also true, because in this case \( F \) is homotopy equivalent to a bouquet of circles.

We first prove Lemma 5.1.B for \( F \neq S^2, \mathbb{RP}^2, T^2, K \) and then separately for the cases \( F = S^2, \mathbb{RP}^2, T^2, K \).

5.3.I. Case \( F \neq S^2, T^2, \mathbb{RP}^2, K \). Corollary 5.3.F says, that \( \pi_1(F, \xi) = Z(\xi) < \pi_1(STF, \xi(a)) \). The corresponding isomorphism (see Section 5.3.C) maps \( q \in \pi_1(F, \xi) \) to \( t_\alpha(q) \in \pi_1(STF, \xi(a)) \) (the lifting by \( h \) of the trajectory of \( a \) under \( q \)).
Thus, for any \( q \in \pi_1(\mathcal{F}, \xi) \) the elements \( t_\alpha(q) \) and \( \tilde{\xi} \) commute in \( \pi_1(\mathcal{STF}, \tilde{\xi}(a)) \). Hence, \( \xi = \text{pr}_*(\tilde{\xi}) \) commutes with \( \text{pr}_*(t_\alpha(q)) \) in \( \pi_1(\mathcal{F}, \xi(a)) \). Proposition 5.3.G implies, that there is an infinite cyclic subgroup of \( \pi_1(\mathcal{F}, \xi(a)) \) generated by some \( g \in \pi_1(\mathcal{F}, \xi(a)) \), which contains both of these loops. Then there exist \( m, n \in \mathbb{Z} \) such that \( \xi = g^m \) and \( \text{pr}_*(t_\alpha(q)) = g^n \).

Consider a curve \( l \) (direct tangent to \( \xi \) at \( \xi(a) \)) which represents \( g \). We can lift it to an element \( \tilde{g} \in \pi_1(\mathcal{STF}, \tilde{\xi}(a)) \).

The kernel of \( \text{pr}_* \) is generated by \( f \), the class of an oriented fiber. Using (5.1) and (5.2) one can interchange \( f \) with the other elements of \( \pi_1(\mathcal{STF}, \tilde{\xi}(a)) \). We get that \( \tilde{g} = \tilde{g}^j f \) and \( t_\alpha(q) = \tilde{g}^i f, \) for some \( k, l \in \mathbb{Z} \). We prove Lemma 5.1.B separately for cases \( m \neq 0 \) and \( m = 0 \) in Section 5.3.J and Section 5.3.K, respectively.

**5.3.J. Case** \( m \neq 0 \).

Remark 5.3.A says that to prove Lemma 5.1.B it is sufficient to show, that \( \Delta_{55}(q^m) \neq 0 \).

One can show that \( t_\alpha(q^m) = \tilde{\xi}^h f^j \) for some \( j \in \mathbb{Z} \). For \( g \) which is an orientation reversing loop follows from the following calculation (which uses (5.1)):

\[
t_\alpha(q^m) = (t_\alpha(q))^m = (\tilde{g}^j f^i)^m = (\tilde{g}^j f^i)^m f^{lm-nk} = \tilde{\xi}^h f^{im-nk}. \tag{5.3}
\]

For \( g \), which is an orientation reversing loop on \( F \), this follows from the similar calculation (which uses (5.2)). The fact that \( t_\alpha(q^m) \) should commute with \( \tilde{\xi} \)

\[
(\text{since it is the } m\text{-th power of } t_\alpha(q) \in Z(\tilde{\xi})) \text{ and the identity } (5.2) \text{ imply, that } j = 0, \text{ provided that } \xi \text{ represents an orientation reversing loop on } F.
\]

Let \( \gamma_1 \) be the sliding of a kink along \( \xi \) (see 2.2.A).

If \( \xi \) represents an orientation reversing loop on \( F \), then the velocity vector of \( \xi \) at \( a \) is rotated by \( 2\pi \) under \( \gamma_1 \). Thus, \( t_\alpha(\gamma_1) = f \), Hence, the loop \( \alpha \in \pi_1(\mathcal{F}, \xi) \) for which \( t_\alpha(a) = t_\alpha(q^m) \) is: \( n \) times sliding of \( \xi \) along itself according to the orientation, composed with \( \gamma_1^n \).

As it was said above, if \( \xi \) represents an orientation reversing loop, then \( t_\alpha(q^m) = \tilde{\xi}^n \). Hence, the loop \( \alpha \in \pi_1(\mathcal{F}, \xi) \), for which \( t_\alpha(a) = t_\alpha(q^m) \) is: \( n \) times sliding of \( \xi \) along itself.

Note, that \( \gamma_1 = 1 \in \pi_1(\mathcal{F}, \xi) \) for \( \xi \) representing an orientation reversing loop on \( F \). To show this one checks, that during the sliding of the kink the velocity vector of \( \xi \) at \( a \) is rotated first by \( 2\pi \) and then by \( (-2\pi) \). (For such \( \xi \) the kink has to slide two times along \( \xi \) before it returns to its original position.) Thus, \( t_\alpha(\gamma_1) = 1 \in \pi_1(\mathcal{STF}, \tilde{\xi}(a)) \) and hence \( \gamma_1 = 1 \in \pi_1(\mathcal{F}, \xi) \).

No triple points appear during the slide of \( \xi \) along itself. Without a loss of generality we can assume that \( \xi \) is the loop for which statement II of Theorem 2.2.C is true. Thus, the inputs of a triple point stratum crossings which occur under \( \gamma_1 \) cancel out. Hence, \( \Delta_{55}(q^m) = 0 \).

Thus, we proved (see 5.3.A) Lemma 5.1.B for \( F \neq S^2, \mathbb{R}P^2, T^2, K \) and \( m \neq 0 \). This case corresponds to \( \xi \neq 1 \in \pi_1(\mathcal{F}, \xi(a)) \).

**5.3.K. Case** \( m = 0 \).

If \( m = 0 \), then \( \xi \) represents \( 1 \in \pi_1(\mathcal{F}, \xi(a)) \). For any \( q \in \pi_1(\mathcal{F}, \xi) \) the projection of \( t_\alpha(q^2) \subset \mathcal{STF} \) to \( F \) is an orientation preserving loop on \( F \). A straightforward check shows that for any \( q \in \pi_1(\mathcal{F}, \xi) \) the element \( q^2 \)
can be obtained by a composition of $\gamma_1^{\pm 1}$ (see 2.2.A) and loops obtained by the following construction.

Push $\xi$ into a small disc by a generic regular homotopy $r$. Slide this small disc along some orientation preserving curve in $F$ and return $\xi$ to its original shape along $r^{-1}$.

Clearly, the inputs of $r$ and $r^{-1}$ into $\Delta_{\mathcal{S}_1}$ cancel out and no triple point stratum crossings happen, when we slide a small disc along a path in $F$. Thus, loops obtained by this construction do not give any input to $\Delta_{\mathcal{S}_1}$. Because of the reasons described in 5.3.J $\Delta_{\mathcal{S}_1}(\gamma_1)$ is also zero.

This implies that for any $g \in \pi_1(F, \xi)$, $\Delta_{\mathcal{S}_1}(g^p) = 0$, and we proved (see 5.3.A) Lemma 5.1.B for $F \neq S^2, \mathbb{RP}^2, T^2, K$.

5.3.L. Case $F = S^2$. One checks that $\pi_1(STS^2) = \mathbb{Z}_2$. Proposition 5.3.F implies that $\pi_1(F, \xi) = \mathbb{Z}_2$ for $F = S^2$. Thus, $\Delta_{\mathcal{S}_1}(g^p) = \Delta_{\mathcal{S}_1}(1) = 0$ (here $1$ is a trivial loop in $F$). This finishes (see 5.3.A) the proof of Lemma 5.1.B for $F = S^2$.

5.3.M. Case $F = T^2$. Using identity (5.1) we get that $\pi_1(STT^2) = Z \oplus Z \oplus Z$. Proposition 5.3.F implies that $\pi_1(F, \xi) = \pi_1(STT^2) = Z \oplus Z \oplus Z$. The generators of this group are:

1) The loop $\gamma_1$, which is the sliding of a kink along $\xi$ (see 2.2.A).
2) The loops $\gamma_2$ and $\gamma_3$, which are slidings of $\xi$ along the unit vector fields parallel to the meridian and the longitude of $T^2$, respectively.

Without the loss of generality we can assume, that $\xi$ is the loop for which statement II of Theorem 2.2.C is true. Thus, $\Delta_{\mathcal{S}_1}(\gamma_1) = 0$. Since no triple point stratum crossing occur during $\gamma_2$ and $\gamma_3$ we get, that $\Delta_{\mathcal{S}_1}(\gamma_2) = \Delta_{\mathcal{S}_1}(\gamma_3) = 0$. This finishes the proof of Lemma 5.1.B for $F = T^2$.

5.3.N. Case $F = \mathbb{RP}^2$. One checks that $\pi_1(ST\mathbb{RP}^2) = \mathbb{Z}_4$. Proposition 5.3.F implies that $\pi_1(F, \xi) = \mathbb{Z}_4$ for $F = \mathbb{RP}^2$. Thus, $\Delta_{\mathcal{S}_1}(g^p) = \Delta_{\mathcal{S}_1}(1) = 0$ (here $1$ is a trivial loop in $F$). This finishes (see 5.3.A) the proof Lemma 5.1.B for $F = \mathbb{RP}^2$.

5.3.O. Case $F = K$. Proposition 5.3.F says, that the group $\pi_1(F, \xi)$ is isomorphic to $Z(\xi)$.

Consider $K$ as a quotient of a rectangle modulo the identification on its sides, which is shown in Figure 6. We can assume, that $\xi(\alpha)$ coincides with the image of a corner of the rectangle, and $\xi$ is direct tangent to the curve $c$ at $\xi(\alpha)$. Let $g$ and $h$ be the curves such that: $\xi(c) = \bar{g}(a) = \bar{h}(a), g = c \in \pi_1(K, L(a))$ and $h = d \in \pi_1(K, L(a))$. (Here $c$ and $d$ are the elements of $\pi_1(K)$ realized by the images of the sides of the rectangle used to construct $K$, see Figure 6.) Let $f$ be the class of an oriented fiber of the fibration $pr : STK \to K$. One can show that:

$$\pi_1(STK, \xi(\alpha)) = \{g, h, f | \bar{h}g^{\pm 1} = \bar{g}f^{\pm 1}h, \bar{h}f^{\pm 1} = f^{\mp 1}h, \bar{g}f = f\bar{g}\}.$$ (5.4)

The second and the third relations in this presentation follow from (5.1) and (5.2). To get the first relation one notes that the identity $dc^{\pm 1} = c^{-1}d \in \pi_1(K, \xi(\alpha))$ implies $\bar{h}g^{\pm 1} = \bar{g}f^{\pm 1}h$ for some $k \in Z$. But $\bar{h}$ commutes with $\bar{g}$,
since they can be lifted to $STT^2$, the fundamental group of which is Abelian. Hence, $k = 0$.

Using relations (5.4) one can calculate $\pi_1^f(\xi, \xi) = \pi_1(f, \xi)$. (Note that these relations allow one to present any element of $\pi_1(STK, \xi(a))$ as $\theta^k\lambda^if^m$, for some $k, l, m \in \mathbb{Z}$.)

A straightforward (but long) check (which uses (5.4)) shows that:

a) If $\xi$ represents an orientation preserving loop on $K$, then a certain degree of any loop $\gamma \in \pi_1(\xi, \xi)$ can be expressed as a product of $\gamma_1$ (see 2.2.A), $\gamma_2$, $\gamma_3$, described below, and their inverses.

b) If $\xi$ represents an orientation reversing loop on $K$, then a certain degree of any loop $\gamma \in \pi_1(\xi, \xi)$ can be expressed as a product of $\gamma_3, \gamma_4$, described below, and their inverses.

To get $\gamma_2$, consider an orientation covering $p : T^2 \to K$. There is a loop $\alpha$ in the space of all autodiffeomorphisms of $T^2$, which is the sliding of $T^2$ along the unit vector field parallel to the lifting of $c$ (see Figure 6). We can assume, that $\alpha$ agrees with the covering structure of $p$. The loop $\gamma_2$ is the projection of the sliding of $p^{-1}(\xi)$, induced by $\alpha$.

Consider a loop $\beta$ in the space of all the autodiffeomorphisms of $K$, which is a sliding of $K$ along the unit vector field parallel to the curve $d$ on $K$. (Note that $K$ has to slide twice along itself under this loop before it comes to its original position.) The loop $\gamma_2$ is the sliding of $\xi$ induced by $\beta$.

The loop $\gamma_4$ is a sliding of $\xi$ along itself.

No triple point stratum crossings occur under $\gamma_2, \gamma_3$ and $\gamma_4$. (For the loop $\gamma_2$ this statement is a little bit less trivial.) Without the loss of generality we can assume, that $\xi$ is the curve for which statement II of Theorem 2.2.C is true. Thus, $\Delta_{\text{str}}(\gamma_i) = 0$ ($i \in \{1, 2, 3, 4\}$) and we proved (see 5.3.A) Lemma 5.1.B for $F = K$.

This finishes the proof of Lemma 5.1.B. \qed

5.4. Proof of Theorem 2.3.B. Let $S$ be the space of all the smooth mappings (not necessarily immersions) from $S^1$ to $F$. Consider the subspace $\mathcal{N}$ of $S^1(3) \times S$ consisting of $t \times f$, such that $f$ maps the three points from $t$ to one point on $F$. Clearly $\mathcal{M}$ is a subspace of $\mathcal{N}$.

Let $s_1, s_2 \in C$ be two generic curves with a triple point, such that $[s_1] = [s_2]$. Let $m_1, m_2 \in \mathcal{M}$ be elements corresponding to $s_1$ and $s_2$. To prove the Theorem
we need to show that $m_1$ and $m_2$ belong to the same path connected component of $\mathcal{M}$.

Since $[s_1] = [s_2]$, we see that $m_1$ can be transformed to $m_2$ (in $\mathcal{M}$) by the sequence of moves $S_1, S_2, S_3, S_4, S_5$ (shown in Figure 7) and their inverses (and a change of the parameterization). Note that $S_1^{\pm 1}$ is the only move in this list which happens not in $\mathcal{M}$.

We can imitate the $S_1$-move staying in $\mathcal{M}$ by creating two opposite kinks (see Figure 2) and then making one of these kinks very small. (Similarly, we can imitate the $S_1^{-1}$-move staying in $\mathcal{M}$.)

![Diagram of moves](image)

**Figure 7**

We use $S_2, S_3, S_4, S_5$ and their inverses and the imitations of $S_1^{\pm 1}$ to deform $m_1$ inside $\mathcal{M}$ so that it looks exactly as $m_2$, except some number of small extra kinks located on the three loops of $m_1$ adjacent to the triple point.

As it is shown below, we can create two opposite extra kinks, the first on one of the three loops of $m_1$, the second on another.

The order in which a small loop going around the triple point crosses the three branches of $m_1$, which pass through the triple point, induces a cyclic order on the branches. We use $S_2$ to deform Figure 8a to Figure 8b, then we use $S_5$ to deform it to Figure 8c. Note, that under this procedure the branches $I$ and $II$ get interchanged in the cyclic order. Then in a similar way we interchange the branches in the pairs $\{I, III\}$, $\{I, II\}$ and $\{I, III\}$. After this the local picture around the triple point is the same as before. One can check, that what happened with $m_1$ globally is equivalent to the addition of two opposite kinks, the first to one branch of $m_1$, the second to another.
It is clear, that using this procedure and the cancelation of the two opposite kinks (see Figure 2) we can concentrate all the extra kinks on one of the three loops of \( m_1 \). Slide these extra kinks along the loop, so that they are all concentrated on a small arc. Cancel out all the pairs of opposite extra kinks (by reversing the process, shown in Figure 2). Now all the small extra kinks are pointing to one side of the loop.

The Hirsch-Smale principle (see [7] page 16) implies that the space of all the loops on \( F \) is weak homotopy equivalent to the space of all the free loops in \( STF \) (the spherical tangent bundle of \( F \)). The corresponding mapping \( h \) sends an immersion \( \xi \in \mathcal{F} \) to a loop \( \tilde{\xi} \subset STF \) by mapping a point \( y \in S^1 \) to a point in \( STF \), to which points the velocity vector of \( \xi \) at \( \xi(y) \). Since by our assumption both \( s_1 \) and \( s_2 \) belong to the same connected component of \( \mathcal{F} \), we get that their liftings to loops in \( STF \) are free homotopic.

Let \( f \) be the homotopy class of the fiber of the \( S^1 \)-fibration \( pr : STF \to F \). An extra small kink corresponds under \( h \) to a multiplication by \( f^{\pm 1} \), depending on the side of the loop the kink points to. Let \( n \) be the number of small extra kinks which are present on \( m_1 \).

Fix a point \( a \) on \( S^1 \). We can assume that after the process described above the curves \( s_1 \) and \( s_2 \) (corresponding to \( m_1 \) and \( m_2 \), respectively) are direct tangent at the image of \( a \). Now we can consider \( s_1 \) and \( s_2 \) as elements of \( \pi_1(F, s_1(a)) \) and the liftings \( \tilde{s}_1 \) and \( \tilde{s}_2 \) (see 5.3.C) as elements of \( \pi_1(STF, \tilde{s}_1(a)) \). By the initial assumption \( s_1 \) and \( s_2 \) belong to the same connected component of \( \mathcal{F} \). The Hirsch-Smale principle implies, that \( \tilde{s}_1 \) is free homotopic to \( \tilde{s}_2 \). Hence, we get that for some element \( \alpha \in \pi_1(STF, s_1(a)) \)

\[
\tilde{s}_1 = \alpha \tilde{s}_1 f^n \alpha^{-1}.
\]  \hspace{1cm} (5.5)

Consider the case of \( F = S^2 \). One checks that \( \pi_1(STS^2) = \mathbb{Z}_2 \) is commutative and \( f \) has order two in \( \pi_1(STS^2) \). From (5.5) we get that \( n \) (the number of extra kinks) is even. We take one of the kinks and evert it, by expanding it till it goes around \( S^2 \) and comes back as a kink pointing to the other side of the loop. Then
we cancel it out with one of the other extra kinks. In order to deform \( m_1 \) to \( m_2 \) we perform this operation until there are no extra kinks left.

In the case of \( F = T^2 \) the group \( \pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) is commutative. From (5.5) we get that \( f^n = 1 \). But \( f \in \pi_1(STT^2) \) has infinite order, thus \( n = 0 \) and there were no extra kinks that survived the process. This means, that we have constructed the desired path from \( m_1 \) to \( m_2 \).

For \( F \neq S^3, T^2 \) the element \( f \in \pi_1(STF) \) has infinite order. Combining identities (5.5) and (5.1) (recall that \( F \) was assumed to be orientable) we get that 
\[
\tilde{g}_1^{-1} \alpha^{-1} \tilde{g}_1 \alpha = f^n.
\]
Thus, the projections of \( \tilde{g}_1 \) and \( \alpha \) commute in \( \pi_1(F, s_1(a)) \). Proposition 5.3.G implies that these projections can be expressed as powers of some \( g \in \pi_1(F, s_1(a)) \). Let \( g_{s_1} \) be a curve representing this \( g \), which is direct tangent to \( s_1 \) at \( s_1(a) \). The kernel of the homomorphism \( \text{pr}_* \) is generated by \( f \). Using identity (5.1) we get, that we can present \( \alpha \) as \( \tilde{g}_1^i f^j \in \pi_1(STF, \tilde{s}_1(a)) \) and \( s_1 \) as \( \tilde{g}_2^k f^l \in \pi_1(STF, \tilde{s}_1(a)) \), for some \( i, j, k, l \in \mathbb{Z} \). But this means (see (5.1)) that \( \alpha \) commutes with \( s_1 \) in \( \pi_1(STF, \tilde{s}_1(a)) \). From the identity (5.5) we get that \( f^n = 1 \). But \( f \) has infinite order in \( \pi_1(STF) \). Hence, \( n = 0 \) and there were no extra kinks, that survived the process. This means that we have constructed the desired path from \( m_1 \) to \( m_2 \). □

5.5. Proof of Theorem 3.2.B. The proof of Theorem 3.2.B is analogous to the proof of Theorem 2.2.C.

One can easily formulate and prove the corresponding versions of Lemma 5.1.A and Lemma 5.1.B.

The strata one has to consider, to prove the analogue of Lemma 5.1.A, are: a) two selftangency points, b) triple point at which exactly two branches are tangent (of order one), c) selftangency point of order two. The bifurcation diagrams for the last two cases are shown in Figure 5 and Figure 9, respectively. □

5.6. Proof of Theorem 3.3.B. The proof of this Theorem is analogous to the proof of Theorem 2.3.B.

Let \( s_1, s_2 \in C \) be two generic curves with a point of direct selftangency (see 3.1.A), such that \( \left[ s_1^+ \right] = \left[ s_2^+ \right] \). Let \( m_1, m_2 \in \mathcal{M}^+ \) be elements corresponding
to \(s_1\) and \(s_2\), respectively. Since \([s_1^+] = [s_2^+]\) we can choose the mappings of \(B_2\) to \(STF\) associated with \(s_1\) and \(s_2\) so that they are free homotopic. Hence, the projections of them to \(F\) are also free homotopic. One can show, that the projections of the two circles of \(B_2\) can be assumed to be direct tangent (at the base point) under this homotopy. Clearly the only moves needed for this homotopy are \(S_1, S_2, S_3\) (see Figure 7), \(S_4, S_5, S_6, S_7\) (see Figure 10) and their inverses.

We use the imitation of \(S_1\) (described in 5.4) \(S_2, S_3, S_4, S_5, S_6, S_7\) and their inverses (and a change of the parameterization) to deform \(m_1\) in the space \(M^+\) to an element, which looks nearly as \(m_2\), except a number of small extra kinks located on the two loops of \(m_1\) adjacent to the point of a direct self-actiongancy.

![Figure 10](image)

For each of the two loops we slide all the extra kinks, so that they are located on a small arc of the loop. We cancel all the pairs of opposite kinks by reversing the process shown in Figure 2. Now the kinks on each loop are pointing to the same side of it.

We note, that the \(T^+\)-equivalence corresponding to \(m_1\) did not change under all these deformations. An extra kink located on a loop of \(m_1\) adds \(f\) (the homotopy class of the fiber of \(pr : STF \to F\)) to the lifting of the loop to a loop in \(STF\). Similarly to 5.4 we get, that for \(F \neq S^2\), the number of extra kinks on each of the two loops of \(m_1\) is zero. This means that we have constructed the desired path connecting \(m_1\) to \(m_2\). For \(F = S^2\) we use the process described in Section 5.4 to cancel out all the extra kinks on each of the two loops, and obtain a path connecting \(m_1\) to \(m_2\).

This finishes the Proof of Theorem 3.3.B. \(\square\)
II. Invariants of Wave Fronts on Surfaces

6. Invariants of Planar Fronts

6.1. Basic facts and definitions. A coorientation of a smooth hypersurface in a functional space is a local choice of one of the two parts, separated by this hypersurface, in a neighborhood of any of its points. This part is called positive.

A contact element on the manifold is a hyperplane in the tangent space to the manifold at a point.

For a (smooth) surface \( F \) we denote by \( STF \) the space of all the cooriented (transversally oriented) contact elements of \( F \). This space is a spherical tangent bundle of \( F \). Its natural contact structure is a distribution of hyperplanes given by a condition, that a velocity vector of an incidence point (on \( F \)) of a contact element belongs to the element. We denote by \( CSTF \) the space of directions in the planes of the contact structure of the manifold \( STF \).

A Legendrian curve \( l \) in \( STF \) is a \((C^1\)-smooth\) immersion of an oriented (circle) \( S^1 \) to \( STF \), such that the velocity vector of \( l \) at every point lies in the plane of the contact structure. We denote by \( \mathcal{M} \) the space of all the Legendrian curves in \( STF \).

The Hirsch-Smale h-principle proved for the Legendrian curves by M. Gromov [7] says, that \( \mathcal{M} \) is weak homotopy equivalent to the space \( \Omega CSTF \) of all the free loops in \( CSTF \). The equivalence is given by the mapping \( h : \mathcal{M} \to \Omega CSTF \), which sends a point on \( S^1 \) (parameterizing a Legendrian curve) to the direction of the velocity vector of the curve at the point.

For a Legendrian curve \( l \) we denote by \( L \) the corresponding wave front, which is a projection of \( l \) to \( F \) equipped with the natural coorientation and orientation. We denote by \( \mathcal{L} \) the space of all the cooriented oriented wave fronts on \( F \). A (cooriented oriented) wave front on \( F \) can be naturally lifted to a Legendrian curve in \( STF \), by mapping a point of it to the direction of the coorienting normal at the point. A generic wave front has only transversal double points and cusp points (of semicubical selftangency) as its singularities.

The fronts which are non generic form a discriminant hypersurface in \( \mathcal{L} \), or for short, the discriminant.

6.1.A. Theorem (Arnold [4]). The strata of codimension one in the dis-
criminant of $\mathcal{C}$ are (see Figure 11):

1) A triple point of a front with pairwise transverse tangent lines at it. This stratum is denoted by $T$ and is called a triple point stratum.

2) A selftangency point of order one of a front. This stratum is denoted by $K$ and is called a selftangency stratum.

3) A cusp point of a front passing through a branch. (Here it is assumed that the line tangent to the front at the cusp point is transversal to the branch.) This stratum is denoted by $\Pi$ and is called a cusp crossing stratum.

4) A singular point of the degree $\frac{4}{3}$. (This is a moment of birth of two cusp points.) This stratum is denoted by $\Lambda$ and is called a cusp birth stratum.

\[ \xymatrix{ \rule{0ex}{2.5ex} \\ \text{cusp birth stratum} } \]
\[ \xymatrix{ \rule{0ex}{2.5ex} \\ \text{selftangency stratum} } \]
\[ \xymatrix{ \rule{0ex}{2.5ex} \\ \text{cusp crossing stratum} } \]
\[ \xymatrix{ \rule{0ex}{2.5ex} \\ \text{triple point stratum} } \]

Figure 11

6.1.B. Whitney and Maslov indices. The Whitney index of an (oriented cooriented) planar wave front is the total rotation number of the coorienting normal
vector of the front. Maslov index of a generic planar wave front is the difference between the number of positive and negative cusps. A cusp is said to be positive, if the branch of the front, which goes away from the cusp belongs to the coorienting half-plane. A cusp is said to be negative, otherwise. (Examples of cusps with different signs are shown in Figure 12.)

Whitney and Maslov indices of a front $L$ are denoted by $\omega(L)$ and $\mu(L)$, respectively. Both this indices do not change under a regular homotopy of a front, which is the projection of a $C^1$-smooth homotopy in the class of Legendrian curves in $ST^*\mathbb{R}^2$. (Note, that the Maslov index is well defined for a wave front on an arbitrary surface $F$.)

![Figure 12](image)

Positive cusps

Negative cusps

The following theorem can be found in [5].

6.1.C. Theorem. Two cooriented oriented planar wave fronts $L_1$ and $L_2$ can be transformed to each other by a regular homotopy if and only if $\omega(L_1) = \omega(L_2)$ and $\mu(L_1) = \mu(L_2)$.

6.2. Invariants $J^+$, $J^-$ and $St'$. 

6.2.A. Definition of sign of a crossing of $K$ stratum. A selftangency of a front is called direct, if the velocity vectors of the two tangent branches have the same direction. A selftangency point is called inverse, otherwise. The $K$-stratum is decomposed into direct selftangency and inverse selftangency parts.

A transversal crossing of a direct selftangency part of the $K$ stratum is said to be positive, if it increases (by two) the number of the double points of the front. It is called negative, otherwise.

A transversal crossing of an inverse selftangency part of the $K$ stratum is said to be positive, if it decreases (by two) the number of the double points of the front. It is called negative, otherwise.

6.2.B. Definition of sign of a crossing of $T$ stratum. Consider transversal crossing of a triple point stratum of the discriminant. A vanishing triangle
is the triangle formed by the three branches of the front, corresponding to a subcritical or to a supercritical value of the parameter near the triple point of a critical front.

The sign of a vanishing triangle is defined by the following construction. The orientation of the front defines the cyclic order on the sides of the vanishing triangle. (It is the order of the visits of the triple point by the three branches.) Hence, the sides of the triangle acquire orientations induced by the ordering. But each side has also its own orientation, which may coincide, or not, with the orientations defined by the ordering.

For each vanishing triangle we define a quantity \( q \in \{0, 1, 2, 3\} \) to be the number of sides of the vanishing triangle equally oriented by the ordering and their direction. The sign of the vanishing triangle is \((-1)^q\).

The sign of the transversal crossing of a triple point stratum is put to be the sign of the new born vanishing triangle.

6.2.C. Definition of Sign of a Crossing of \( \Pi \) Stratum. Consider transversal crossing of a cusp crossing stratum. Substitute a cusp point by a small kink (see Figure 13). We put the sign of a transversal crossing of the \( \Pi \) stratum to be equal to the sign of the new-born vanishing triangle, which appears under the crossing of the branch by the kink. (The idea of substitution of a cusp by a kink appeared in the work of M. Polyak [10].)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig13}
\caption{}
\end{figure}

6.2.D. Definition. A selftangency of a wave front is called a dangerous selftangency, if the coorientations of the tangent branches coincide; and a safe selftangency, otherwise. (A front with a point of dangerous selftangency lifts to a Legendrian knot with a double point.) This relation induces a subdivision of the stratum \( K \) into the strata \( K^+ \) and \( K^- \) of respectively dangerous and safe selftangencies.

6.2.E. Theorem (Aicardi [1], Arnold [5], Polyak [10]).

There exist three numbers \( St(L), J^+(L), J^-(L) \) assigned to a generic planar wave front \( L \), which are uniquely defined by the following properties.
1. \( \text{St}'(L) \), \( J^+(L) \) and \( J^-(L) \) are invariant under a regular homotopy in the class of generic planar wave fronts.

2. \( \text{St}'(L) \) does not change under a crossing of selftangency and cusp-birth strata, increases by one under a positive triple point stratum crossing and by \( \frac{1}{2} \) under a positive crossing of a cusp crossing stratum.

3. \( J^+(L) \) does not change under a crossing of a triple point, cusp crossing, cusp birth and safe selftangency strata and increases by two under a positive crossing of a dangerous selftangency stratum.

4. \( J^-(L) \) does not change under a crossing of a triple point, cusp crossing, cusp birth and dangerous selftangency strata and increases by two under a positive crossing of a safe selftangency stratum.

5. On the standard fronts \( K_{\omega,k} \), shown in Figure 14, \( \text{St}'(L) \), \( J^+(L) \) and \( J^-(L) \) take the following values (independent on the choice of an orientation and coorientation of the standard fronts):

\[
\text{St}'(K_{0,k}) = \frac{k}{2}, \quad \text{St}'(K_{\omega+1,k}) = \omega + \frac{k}{2} \ (\omega = 0, 1, 2, \ldots) \quad (6.1)
\]

\[
J^+(K_{0,k}) = -k, \quad J^+(K_{\omega+1,k}) = -2\omega - k \ (\omega = 0, 1, 2, \ldots) \quad (6.2)
\]

\[
J^-(K_{0,k}) = -1, \quad J^-(K_{\omega+1,k}) = -3\omega \ (\omega = 0, 1, 2, \ldots) \quad (6.3)
\]

where \( k = 0, 1, 2, \ldots \).

---

\[\quad\]

\( K_{0,k} \)

---

\( K_{1,k} \)

---

\( K_{\omega,k} \)

---

**Figure 14**
7. Invariants of Fronts on Orientable Surfaces

7.1. Natural decomposition of a dangerous selftangency point stratum.

7.1.A. Definition. Let $F$ be a surface. We say that a front $L \subset F$ with a dangerous selftangency point $q$ is a generic front with a dangerous selftangency point, if its only nongeneric singularity is this point.

7.1.B. Let $F$ be a surface. Let $B_2$ be a bouquet of two oriented circles, and $b$ be its base point. Let $L : S^1 \rightarrow F$ be a generic front with a dangerous selftangency point $q$. It can be lifted to the mapping $\bar{L} : S^1 \rightarrow STF$, which sends a point $p \in S^1$ to the point in $STF$, to which points the coorienting normal of $L$ at $L(p)$. (Note, that $q$ lifts to a double point $\bar{q}$ of $\bar{L}$.) There exists an associated with $\bar{L}$ mapping $\phi : B_2 \rightarrow STF$, which satisfies the following conditions:

   a) $\phi$ is injective and $\phi(B_2) = \bar{L}(S^1)$.
   b) $\phi(b) = \bar{q}$.
   c) $\phi$ is orientation preserving.

Note, that the free homotopy class of a mapping of $B_2$ to $STF$ realized by $\phi$ is well defined, modulo the orientation preserving automorphism of $B_2$ which interchanges the circles.

7.1.C. Definition of $T^+\text{-equivalence}$. Let $L_1$ and $L_2$ be two generic fronts with a point of dangerous selftangency (see 7.1.A). We say that these fronts are $T^+\text{-equivalent}$, if there exist associated with the two of them mappings of $B_2$, which are free homotopic. A dangerous selftangency point stratum is naturally decomposed into parts corresponding to different $T^+\text{-equivalence classes}$.

We denote by $[L^+]$ the $T^+\text{-equivalence class}$ corresponding to $L$, a generic front with a point of dangerous selftangency. We denote by $T^+$ the set of all the $T^+\text{-equivalence classes}$.

7.2. Axiomatic description of $J^\mp$. A natural way to introduce $J^+$ type invariant of generic wave fronts on a surface $F$ is to take a function $\psi : T^+ \rightarrow \mathbb{Z}$ and to construct an invariant of generic wave fronts from a fixed connected component $C$ of the space $L$ (of all the fronts on $F$), such that:

1. It increases by $\psi([L^+])$ under a positive crossing of the part of a dangerous selftangency stratum, which corresponds to a $T^+\text{-equivalence class}$ $[L^+]$.
2. It does not change under a crossing of safe selftangency, triple point, cusp crossing and cusp birth strata of the discriminant.

If for a given function $\psi : T^+ \rightarrow \mathbb{Z}$ there exists such an invariant of wave fronts from $C$, then we say that there exists a $J^\mp$ invariant of fronts in $C$, which integrates this function. Such $\psi$ is said to be $J^\mp\text{-integrable}$ in $C$.

7.2.A. Theorem. Let $F$ be an orientable surface (not necessarily compact), $T^+$ be the set of all the $T^+\text{-equivalence classes}$ and $C$ be a connected component of $L$. Let $\psi : T^+ \rightarrow \mathbb{Z}$ be a function.

Then there exists a unique (up to an additive constant) invariant $J^\mp$ of generic wave fronts in $C$, which integrates $\psi$.

The Proof of this Theorem (see Section 10) is based on Theorem 8.2.A.
Remark. Note, that if $F$ is orientable then a function $\psi : T^+ \to \mathbb{Z}$ is integrable for all connected components of $L$.

However, if $F$ is nonorientable then such an invariant exists not for all functions $\psi$. Moreover, if a function $\psi$ is $J^+$-integrable in one connected component of $L$, it is not necessarily $\bar{J}^+$-integrable in another component of $L$. In Theorem 8.2.A we present a condition on $\psi$, which is necessary and sufficient for it to be $\bar{J}^+$-integrable in a fixed connected component of $L$.

7.2.B. Connection with the standard $J^+$-invariant. Since $\pi_1(S^1 \mathbb{R}^3)$ is isomorphic to $\mathbb{Z}$, there are countably many $T^+$-equivalence classes of singular fronts on $\mathbb{R}^2$, which can be obtained from a front of the fixed Whitney and Maslov indices. (Note, that Whitney and Maslov indices of the planar front $L$ define the connected component of the space of all fronts on $\mathbb{R}^2$, where $L$ belongs to.) Thus, the construction of $\bar{J}^+$ gives rise to a splitting of the standard $J^+$ invariant of V. Arnold. This is the splitting introduced by V. Arnold [4] in the case of planar fronts of the zero Whitney index, and generalized to the case of arbitrary planar wave fronts by F. Aicardi [2].

7.3. Natural decomposition of a safe self-tangency stratum.

7.3.A. Definition. Let $F$ be a surface. We say that a front $L \subset F$ with a safe self-tangency point $q$ is a generic front with a safe self-tangency point, if its only non-generic singularity is this point.

7.3.B. Let $F$ be a surface. Let $B_2$ be a bouquet of two oriented circles, and $b$ be its base point. Let $L : S^1 \to F$ be a generic front with a safe self-tangency point $q$. It can be lifted to the mapping $\bar{L}$ from the oriented circle to $PTF$ (the projectivized tangent bundle of $F$), which sends a point $p \in S^1$ to the point in $PTF$, to which points the normal line to $L$ at $L(p)$. (Note, that $q$ lifts to a double point $\bar{q}$ of $\bar{L}$.) There exists an associated with $L$ mapping $\phi : B_2 \to PTF$, which satisfies the following conditions:

a) $\phi$ is injective and $\phi(B_2) = \bar{L}(S^1)$.

b) $\phi(b) = \bar{q}$.

c) $\phi$ is orientation preserving.

Note, that the free homotopy class of a mapping of $B_2$ to $PTF$ realized by $\phi$ is well defined, modulo the orientation preserving automorphism of $B_2$ which interchanges the circles.

7.3.C. Definition of $T^-$-equivalence. Let $L_1$ and $L_2$ be two generic fronts with a point of safe self-tangency (see 7.3.A). We say that these fronts are $T^-$-equivalent, if there exist associated with the two of them mappings of $B_2$, which are free homotopic. A safe self-tangency point stratum is naturally decomposed into parts corresponding to different $T^-$-equivalence classes.

We denote by $[L^-]$ the $T^-$-equivalence class corresponding to $L$, a generic front with a point of safe self-tangency. We denote by $T^-$ the set of all the $T^-$-equivalence classes.
7.4. Axiomatic description of $J^-$. A natural way to introduce $J^-$-type invariant of generic wave fronts on a surface $F$ is to take a function $\psi : T^- \to \mathbb{Z}$ and to construct an invariant of generic wave fronts from a fixed connected component $C$ of the space $L$ (of all the fronts on $F$), such that:
1. It increases by $\psi([L^-])$ under a positive crossing of the part of a safe selftangency stratum, which corresponds to a $T^-$-equivalence class $[L^-]$.
2. It does not change under a crossing of a dangerous selftangency, triple point, cusp crossing and cusp birth strata of the discriminant.

If for a given function $\psi : T^- \to \mathbb{Z}$ there exists such an invariant of wave fronts in $C$, then we say that there exists a $J^-$-invariant of fronts in $C$, which integrates this function. Such $\psi$ is said to be $J^-$-integrable in $C$.

7.4.A. Theorem. Let $F$ be an orientable surface (not necessarily compact), $T^-$ be the set of all the $T^-$-equivalence classes and $C$ be a connected component of $L$. Let $\psi : T^- \to \mathbb{Z}$ be a function.

Then there exists a unique (up to an additive constant) invariant $J^-$ of generic wave fronts in $C$, which integrates $\psi$.

The proof of this Theorem is analogous to the Proof of Theorem 7.2.A (see Section 10) and is based on Theorem 8.2.A.

Remark. Note, that if $F$ is orientable then a function $\psi : T^- \to \mathbb{Z}$ is integrable for all connected components of $L$.

However, if $F$ is nonorientable then such an invariant exists not for all functions $\psi$. Moreover, if a function $\psi$ is $J^-$-integrable in one connected component of $L$, it is not necessarily $J^-$-integrable in another component of $L$. In Theorem 8.2.A we present a condition on $\psi$, which is necessary and sufficient for it to be $J^-$-integrable in a fixed connected component of $L$.

7.4.B. Connection with the standard $J^-\mathrm{-invariant}$. Since $\pi_1(PTR^2)$ is isomorphic to $\mathbb{Z}$, there are countably many $T^-$-equivalence classes of singular wave fronts on $\mathbb{R}^2$, which can be obtained from a front of the fixed Whitney and Maslov indices. (Note, that Whitney and Maslov indices of the planar front $L$ define the connected component of the space of all fronts on $\mathbb{R}^2$, where $L$ belongs to.) Thus, the construction of $J^-$ gives rise to a splitting of the standard $J^-$-invariant of V. Arnold. This splitting is analogous to the splitting of $J^+$ introduced by V. Arnold [4] in the case of planar fronts of the zero Whitney index, and generalized to the case of arbitrary planar wave fronts by F. Aicardi [2].

7.5. Natural decomposition of a triple point and a cusp crossing strata.

7.5.A. Definition. Let $F$ be a surface. We say that a front $L \subset F$ with a triple point $q$ is a generic front with a triple point, if its only nongeneric singularity is this triple point, at which every two branches are transverse to each other. We also say that a front $L \subset F$ with a cusp crossing point $q$ is a generic front with a cusp crossing point, if its only nongeneric singularity is this cusp crossing point, at which the lines tangent to the two branches of the front are transverse.
7.5.B. Let $F$ be an oriented surface. Let $B_3$ be a bouquet of three oriented circles, with a fixed cyclic order on them, and $b$ be the base point of $B_3$. Let $L : S^1 \to F$ be a generic front with a triple point $q$. There exists an associated with $L$ mapping $\phi : B_3 \to F$, which satisfies the following conditions:

a) $\phi(B_3) = L(S^1)$.

b) $\phi(b) = q$.

c) $\phi$ is injective on the preimage of the complement of the multiple points of the front $L$.

d) $\phi$ is orientation preserving.

e) The cyclic order, induced on the circles of $B_3$ by traversing $L(S^1)$ according to the orientation, coincides with the fixed one.

Note, that the free homotopy class of the mapping of $B_3$ to $F$ realized by $\phi$ is well defined, modulo an automorphism of $B_3$ which preserves the orientation and the cyclic order on the circles.

7.5.C. Definition of $T$-equivalence. Let $s_1$ and $s_2$ be two generic oriented fronts with a triple point (see 7.5.A). We say, that these curves are $T$-equivalent, if there exist associated with them mappings of $B_3$ which are free homotopic. A triple point stratum is naturally decomposed into parts corresponding to different $T$-equivalence classes.

Amazingly enough, the $T$-equivalence relation induces also a subdivision of a cusp crossing stratum of the discriminant. To see it one substitutes the cusp on a generic front $L$ with a cusp crossing point by a small kink, as it is shown in Figure 13. As a result of this operation the front changes to a generic non-coorientable front $L'$ with a triple point. We take the $T$-equivalence class corresponding to the front $L$ to be the $T$-equivalence class corresponding to the front $L'$.

We denote by $[L]$ the $T$-equivalence class corresponding to $L$, a generic front with a triple point or with a cusp crossing point. We denote by $T$ the set of all the $T$-equivalence classes.

7.6. Axiomatic description of $\overline{ST'}$. A natural way to introduce $ST'$ type invariants of generic wave fronts on a surface $F$ is to take a function $\psi : T \to Z$ and to construct an invariant of generic wave fronts from a fixed connected component $C$ of the space $\mathcal{C}$ (of all the fronts on $F$) such that:

1. It does not change under a crossing of selftangency or cusp birth strata of the discriminant.

2. It increases by $\psi([L])$ under a positive crossing of the part of a triple point stratum, which corresponds to a $T$-equivalence class $[L]$.

3. It increases by $\frac{1}{2} \psi([L])$ under a positive crossing of the part of a cusp crossing stratum, which corresponds to a $T$-equivalence class $[L]$. (As it is shown in section 9.1, one can not substitute $\frac{1}{2}$ by another constant and construct an invariant of this sort, unless $\psi$ is put to be identically zero on all the $T$-equivalence classes, which appear on a cusp crossing stratum.)

If for a given function $\psi : T \to Z$ there exists such an invariant of wave fronts from $C$, then we say that there exists an $\overline{ST'}$ invariant of fronts in $C$, which integrates this function. Such $\psi$ is said to be $\overline{ST'}$-integrable in $C$. 
Such an invariant exists not for all functions $\psi$. In Theorem 8.2.A we present a condition on $\psi$, which is necessary and sufficient for the existence of such an invariant.

If the case of orientable $F$ there is a simple condition on $\psi$, which is sufficient for the existence of such an invariant.

7.6.A. **Theorem.** Let $F$ be an orientable surface (not necessarily compact), $T$ be the set of all the $T$-equivalence classes and $C$ be a connected component of $C$. Let $\psi : T \to \mathbb{Z}$ be a function, which takes equal values on any two $T$-equivalence classes, such that:

a) The free homotopy classes of the mappings of $B_3$ representing them are different by an orientation preserving automorphism of $B_3$, which changes the cyclic order on the circles.

b) The restrictions of the mapping of $B_3$ representing these classes to one of the circles are homotopically trivial.

Then there exists a unique (up to an additive constant) invariant $\overline{St'}$ of generic wave fronts in $L$, which integrates $\psi$.

The Proof of this Theorem is analogous to the Proof of Theorem 7.2.A (see Section 10) and is based on Theorem 8.2.A.

Remark. Note, that if $F$ is orientable, then a function $\psi : T \to \mathbb{Z}$ described in Theorem 7.6.A is integrable for all the connected components of $L$.

However in general such an invariant exists not for all functions $\psi$. Moreover, if a function $\psi$ is $\overline{St'}$-integrable in one connected component of $L$ it is not necessarily $\overline{St'}$-integrable in another component of $L$. In Theorem 8.2.A we present a condition on $\psi$, which is necessary and sufficient for it to be $\overline{St'}$-integrable in a fixed connected component of $L$.

7.6.B. **Connection with the standard $St'$-invariant.** Since $\pi_1(\mathbb{R}^2) = 0$, there is just one $T$-equivalence class of singular immersions to $\mathbb{R}^2$. Thus, the construction of $\overline{St'}$ does not give anything new in the classical case of planar wave fronts.

8. **Necessary and Sufficient Conditions for the Integrability**

8.1. **Obstructions for the integrability.** Let $\psi : T \to \mathbb{Z}$ be a function and let $\gamma$ be a generic loop in a connected component $C$ of $L$. Let $I_1$ and $I_2$ be the sets of moments, when $\gamma$ crosses respectively a triple point and a cusp crossing strata. Let $\{\sigma_1\}_{i \in I_1}$ and $\{\sigma_i\}_{i \in I_2}$ be the signs of the corresponding crossings of the strata; and let $\{[s_i]\}_{i \in I_1}$ and $\{[s_i]\}_{i \in I_2}$ be the $T$-equivalence classes corresponding to the parts of the strata, in which the crossings occurred. Put

$$\Delta_{\overline{St'}}(\gamma) = \sum_{i \in I_1} \sigma_i \psi([s_i]) + \sum_{i \in I_2} \sigma_i \frac{1}{2} \psi([s_i]). \quad (8.1)$$
II. INVARIANTS OF WAVE FRONTS ON SURFACES

If $\Delta_{ST}(\gamma) = 0$, then $\psi$ is said to be integrable along $\gamma$. We call $\Delta_{ST}(\gamma)$, the change of $\overset{\circ}{\Sigma}^\circ$ along $\gamma$.

In a similar way we define a notion of integrability along $\gamma$ for integer valued functions on $T^+$ and on $T^-$. (For this we use the intersections of $\gamma$ with dangerous and safe self-tangency strata, respectively.) The changes of $J^-$ and of $J^+$ along $\gamma$ are also defined in a similar way.

It is clear, that if a function $\psi$ is integrable in $C$, then it is integrable along any loop $\gamma \subset C$.

In this section we describe two loops $\gamma_1$ and $\gamma_2$ in $C$, such that integrability along them implies integrability in $C$. In a sense, the changes along them are the only obstructions for the integrability.

8.1.A. Loop $\gamma_1$. Let $C$ be a connected component of $L$ and $L \in C$ be a generic wave front. Let $\gamma_1$ be a loop constructed below.

Deform $L$ along a generic path $t$ in $C$ to get two opposite kinks, as it shown in Figure 2. Make the first kink small and move it along the front till it comes back. We require the deformation to be such, that at each moment of time points of $L$ located outside of a small neighborhood of the kink do not move. (In Figure 3 and Figure 15 it is shown how the kink passes a double and a cusp point.)

Finally deform $L$ to its original shape along $t^{-1}$.

Note, that if $L$ represents an orientation reversing loop on $F$, then the kink slides two times along $L$, before it returns to its original position.

8.1.B. Loop $\gamma_2$. Let $C$ be a connected component of $L$ and $L \in C$ be a generic wave front. Let $\gamma_2$ be a loop constructed below.

Deform $L$ along a generic path $t$ in $C$, so that all the cusps are concentrated on a small piece of $L$ and the side of $L$, they are pointing to, alternates. (The notion of side is locally well-defined.) This is possible because, as it is shown in Figure 16, we can cancel a pair of adjacent cusps pointing to the same side of $L$.

If after such a modification the number of cusps is nonzero we freeze the shape of the piece $P$ and slide the shape along $L$ till it comes back. We require the deformation to be such, that at each moment of time points of $L$ located outside of a small neighborhood of the piece $P$ do not move. (Note, that if $L$ represents an orientation reversing loop on $F$, then we have to slide this shape twice around $L$, before it returns to the original position.)
If after such a modification the number of cusps is zero (this happens if \( \mu(L) = 0 \)), then we perform a regular homotopy shown in Figure 17.

During this regular homotopy we fix a point \( a \) on a circle parameterizing \( L \). Under this homotopy the image of a small neighborhood of \( a \) is supposed to be frozen all the time except when two cusps are passing through it before sliding around the wave front.

Finally we deform \( L \) to its original shape along \( t^{-1} \).

We can use different paths \( t \) in this loop. So to make the loop \( \gamma_2 \) well-defined we fix a path \( t \) for each particular wave front \( L \).

\[ \includegraphics[width=0.5\textwidth]{figure16.png} \]

**Figure 16**

8.2. Integrability theorem. Now we are ready to formulate the main integrability theorem.

**8.2.A. Theorem.** Let \( F \) be a surface (not necessarily compact or orientable), \( C \) be a connected component of \( L \) and \( L \in C \) be a generic wave front. Let \( \psi_1: T \to \mathbb{Z}, \psi_2: T^+ \to \mathbb{Z} \) and \( \psi_3: T^- \to \mathbb{Z} \) be functions and \( \gamma_1, \gamma_2 \subset C \) be the loops (starting at \( L \)) described in section 8.1.A and section 8.1.B, respectively. Then:

I: The condition, that \( \psi_2 \) is integrable along loops \( \gamma_1 \) and \( \gamma_2 \) is necessary and sufficient for the existence of a \( \tilde{S}^1 \) invariant, which integrates \( \psi_1 \) in \( C \).

II: The condition, that \( \psi_2 \) is integrable along loops \( \gamma_1 \) and \( \gamma_2 \) is necessary and sufficient for the existence of a \( \tilde{T}^+ \) invariant, which integrates \( \psi_2 \) in \( C \).

III: The condition, that \( \psi_3 \) is integrable along loops \( \gamma_1 \) and \( \gamma_2 \) is necessary and sufficient for the existence of a \( \tilde{T}^- \) invariant, which integrates \( \psi_3 \) in \( C \).

**8.2.B. Remarks to Theorem 8.2.A.** If an invariant from the statement of the Theorem exists, then it is unique up to an additive constant.

The choice of \( L \in C \) does not matter and to check integrability of a given function it is easier to take \( L \in C \) such that:

a) It already has a small kink.

b) All the cusps are already located on a small piece of \( L \) and the (locally well-defined) side of \( L \), they point to, alternates.

Using the approach of this chapter, one can construct generalizations of other local invariants of planar wave fronts, which were studied by F. Aicardi [1].
Figure 17
9. Proof of Theorem 8.2.A

We are going to prove only the first statement of the Theorem 8.2.A. The proofs of the other two statements are obtained in a similar way.

Clearly, in order for $\bar{S}^{j'}(L)$ to be well defined, the sum of the jumps of it under the deformations of the front described by the loops $\gamma_1$ and $\gamma_2$ (see 8.1.A and 8.1.B) has to be zero. Thus, this condition is necessary for the integrability of $\psi_1$. Let us prove, that it is also sufficient for the integrability of $\psi_1$.

Fix a connected component $C$ of the space of all immersions and a generic wave front $L \in C$. Fix any value of $\bar{S}^{j}(L) \in \mathbb{Z}$. Let $L' \in C$ be another generic wave front. Take a generic path $p$ in $C$, which connects $L$ with $L'$. When we go along this path, we see a sequence of crossings of a selftangency, cusp birth, cusp crossing and triple point strata of the discriminant. Let $J_1$ and $J_2$ be the sets of moments, when $p$ crosses respectively a triple point and a cusp crossing stratum. Let $\{\sigma_i\}_{i \in J_1}$ and $\{\sigma_i\}_{i \in J_2}$ be the signs of the corresponding crossings of the strata; and let $\{[s_i]\}_{i \in J_1}$ and $\{[s_i]\}_{i \in J_2}$ be the $T$-equivalence classes corresponding to the parts of the strata, in which the crossings occured. Put $\Delta_{\bar{S}^j}(p) = \sum_{i \in J_1} \sigma_i \psi([s_i]) + \sum_{i \in J_2} \sigma_i \psi([s_i])$ and $\bar{S}^{j'}(L') = \bar{S}^{j}(L) + \Delta_{\bar{S}^j}(p)$.

To prove the Theorem it is sufficient to show, that $\bar{S}^{j}(L')$ does not depend on the generic path $p$, we used to define it. The last statement follows from Lemma 9.0.C and Lemma 9.0.D. Thus, we proved Theorem 8.2.A modulo these two Lemmas. □

9.0.C. Lemma. Let $p$ be a generic path in $C$, which connects $L$ to itself. Then $\Delta_{\bar{S}^j}(p)$ depends only on the element of $\pi_1(C,L)$ realized by $p$.

9.0.D. Lemma. If the function $\psi_1$ is integrable along the loops $\gamma_1$ and $\gamma_2$, then for every element of $\pi_1(C,L)$ there exists a generic loop $q$ in $C$, representing this element, such that $\Delta_{\bar{S}^j}(q) = 0$.

9.1. Proof of Lemma 9.0.C. To prove the Lemma it is sufficient to show that, if we go around any stratum of codimension two along a small generic loop $r$ (not necessarily starting at $L$), then $\Delta_{\bar{S}^j}(r) = 0$. All the codimension two strata are described in the following Theorem.

9.1.A. Theorem (Arnold [4]). All the strata of codimension two in the discriminant of $L$ are (see Figure 18 and Figure 19):

1) A quadruple point of the front with pairwise transverse tangent lines. This stratum is denoted by $TT$.

2) A degenerate triple point, at which two branches are tangent of order one and the third branch is transverse to them. This stratum is denoted by $KT$.

3) A point of a cubical selftangency. This stratum is denoted by $KK$.

4) A cusp passing through a branch in such a way, that they have the same tangent line. This stratum is denoted by $KII$.

5) A cusp point point passing simultaneously through two branches. (Here it is assumed that the lines tangent to the three participating branches are different.) This stratum is denoted by $TII$. 


6) Two coinciding cusp points. (Here it is assumed that the lines tangent to the two branches are different.) This stratum is denoted by \( \Pi \Pi \).

7) A point of degree \( \frac{4}{3} \) passing through a branch of the front. (Here it is assumed that the two branches have transverse tangent lines.) This stratum is denoted by \( \Pi \Lambda \).

8) A point of degree \( \frac{2}{3} \). This stratum is denoted by \( \Lambda \Lambda \).

Also codimension two have the strata of transverse intersection (or selfintersection) of the closures of two strata of codimension one. This corresponds to wave fronts having two points of nongenericity.

Figure 18

9.1.B. The only strata of codimension two, in the bifurcation diagram of which triple points or cusp crossings are present are: two distinct triple points, two distinct cusp points, and strata \( TT \), \( KT \), \( K \Pi \), \( T \Pi \), \( \Pi \Pi \), \( \Pi \Lambda \) and \( \Lambda \Lambda \) (in the notation of Lemma 9.1.A).

If \( \gamma \) is a small loop, which goes around a stratum of two distinct triple points, then in \( \Delta_{\text{ST}}(\gamma) \) we have each of the two \( T \)-equivalence classes twice, once with the plus sign of the newborn vanishing triangle, once with the minus. Hence \( \Delta_{\text{ST}}(\gamma) = 0 \).

The same argument shows, that if \( \gamma \) is a small loop going around a stratum of two distinct cusp points, then \( \Delta_{\text{ST}}(\gamma) = 0 \).

To prove the statement for the other strata we use the bifurcation diagrams shown in Figure 18 and Figure 19.
Let \( r \) be a small loop, which goes around the \( TT \) stratum. We can assume, that it corresponds to a loop in Figure 5 directed counter clockwise. There are eight terms in \( \Delta_{ST}(r) \). We split them into pairs I, II, III, IV, as it is shown in Figure 5. One can see, that the \( T \)-equivalence classes of the two wave fronts in each pair are the same. For each branch the sign of the colored triangle is equal to the sign of the triangle, which died under the triple point stratum crossing shown on the next (in the counterclockwise direction) branch. The sign of the dying vanishing triangle is minus the sign of the newborn vanishing triangle. Finally, one can see that the signs of the colored triangles inside each pair are opposite. Thus, all these eight terms cancel out.

Let \( r \) be a small loop which goes around the \( KT \) stratum. There are just two terms in \( \Delta_{ST}(r) \). It is clear, that the \( T \)-equivalence classes in them coincide. A direct check shows that the signs of the two terms are opposite. (Note, that if they are not opposite, then Arnold's St invariant is not well defined.)

In the same way one shows that \( \Delta_{ST}(r) = 0 \), for \( r \) being a small loop going around the \( K II \) stratum, or around the \( AA \) stratum.

There are four terms in \( \Delta_{ST}(r) \), for \( r \) being a loop going around the \( III \) stratum. We split them into pairs, such that the terms in a pair are created by the same cusp crossing two different branches. One checks that the \( T \)-equivalence classes inside a pair coincide and the signs of the terms are opposite. Thus the terms inside each pair cancel out.

There are six terms in \( \Delta_{ST}(r) \), for \( r \) being a loop going around the \( T II \) stratum. Two of them correspond to \( r \) intersecting a triple point stratum, and four to \( r \) intersecting a cusp crossing stratum. One checks that the two terms corre-
II. INVARIANTS OF WAVE FRONTS ON SURFACES

Corresponding to \( r \) intersecting a triple point stratum cancel out. (As before the two \( T \)-equivalence classes are the same, and the signs with which they participate are opposite.) We split the other four terms into pairs, such that the terms in a pair are created by the cusp crossing the same branch of the front. One checks that the \( T \)-equivalence classes inside a pair coincide and the signs of the terms are opposite. Thus, the terms inside each pair cancel out.

Finally, let \( r \) be a loop going around the \( II \) stratum. There are three terms in \( \Delta_{\text{str}}(r) \). One of them corresponds to \( r \) intersecting a triple point stratum and two, to \( r \) intersecting a cusp crossing stratum. One checks, that the sign of the intersection of a triple stratum by \( r \) is opposite from the signs of the intersections of a cusp crossing stratum, and that all the three \( T \)-equivalence classes are equal. We denote the class by \( t \). Thus, \( \Delta_{\text{str}}(r) = \frac{1}{2} \psi(t) + \frac{1}{2} \psi(t) - \psi(t) = 0 \).

Recall, that the magnitude of the jump of \( St \) under the interaction of the part of the cusp crossing stratum, corresponding to the \( T \)-equivalence class \( t \), was put to be \( \frac{1}{2} \psi(t) \). One can see, that if we substitute \( \frac{1}{2} \) by another constant, then there is no hope for constructing an invariant of this sort, unless \( \psi(t) = 0 \) for every \( T \)-equivalence class \( t \), which appears on a cusp crossing stratum.

This finishes the Proof of Lemma 9.0.C. □

9.2. Constructions and facts needed for the proof of Lemma 9.0.D.

9.2.A. Parametric \( h \)-principle. We will use the following notation: for a surface \( F \) we denote by \( STF \) the spherical tangent bundle of \( F \) and by \( CSTF \) the space of directions in the planes of the natural contact structure of \( STF \) (see section 6.1). For a Legendrian immersed curve \( l \subset STF \) we denote by \( L \) the corresponding wave front on \( F \) and by \( \tilde{l} \) a loop in \( CSTF \) obtained by sending a point of \( l \) to the direction of the velocity vector of \( l \) at the point.

The parametric \( h \)-principle, proved for the Legendrian curves by M. Gromov [7], says that the space of wave fronts \( \mathcal{L} \) is weak homotopy equivalent to the space \( \Omega CSTF \) of all the free loops in \( CSTF \).

The mapping \( h : \mathcal{L} \to \Omega CSTF \), which gives the equivalence, sends a wave front \( L \) (corresponding to the Legendrian curve \( l \)) to the loop \( \tilde{l} \in \Omega CSTF \).

Fix a point \( a \) on \( S^1 \) (which parameterizes the fronts). Let \( q \) be a loop in \( \mathcal{L} \) starting at \( L \). For any moment of time \( q(t) \) is a wave front, which can be lifted to a loop in \( CSTF \). Thus, \( q \) gives rise to the mapping \( \gamma_q : S^1 \times S^2 \to CSTF \) (the lifting of \( q \) by \( h \)). (In the product \( S^1 \times S^2 \) the first copy of \( S^1 \) corresponds to the parameterization of a front and the second to the parameterization of the loop \( q \).) The mapping \( \gamma_q \) restricted to \( a \times S^2 \) gives rise to a loop \( t_a(q) \) in \( CSTF \). (It is a trajectory of a lifting of \( a \).) One can check, that the mapping \( t_a : \pi_1(\mathcal{L}, L) \to \pi_1(CSTF, \tilde{l}(a)) \) is a homomorphism.

9.2.B. Proposition. \( t_a(q) : \pi_1(\mathcal{L}, L) \to \pi_1(CSTF, \tilde{l}(a)) \) is an isomorphism of \( \pi_1(\mathcal{L}, L) \) onto \( Z(\tilde{l}) \), the centralizer of \( \tilde{l} \in \pi_1(CSTF, \tilde{l}(a)) \).

9.2.C. Proof of Proposition 9.2.B. Let \( p : \Omega CSTF \to CSTF \) be the mapping, which sends \( \omega \in \Omega CSTF \) to \( \omega(a) \in CSTF \). (One can check, that this \( p \) is a Serre fibration, with the fiber of it isomorphic to the space of loops based
at the corresponding point.) Because of the h-principle (see 9.2.A) it is clear, that to prove the Proposition it is sufficient to show, that \( p_* : \pi_1(O CSTF, i) \to \pi_1(CSTF, i(a)) \) gives an isomorphism of \( \pi_1(O CSTF, i) \) onto \( Z(i) \).

A Proposition proved by V.L. Hansen [8] says that: if \( X \) is a topological space with \( \pi_2(X) = 0 \), then \( \pi_1(\Omega X, \omega) = Z(\omega) \leq \pi_1(X, \omega(a)) \). (Here \( \Omega X \) is the space of free loops in \( X \) and \( \omega \) is an element of \( \Omega X \).) One can check that \( \pi_2(CSTF) = 0 \) for any surface \( F \). Thus, we get that \( \pi_1(O CSTF, i) \) is isomorphic to \( Z(i) \leq \pi_1(CSTF, i(a)) \). From the proof of the Hansen's Proposition it follows that the isomorphism is induced by \( p_* \). \( \square \)

9.2.D. **Proposition.** Let \( \gamma_1, \gamma_2 \in \pi_1(C, L) \) be the loops described in 8.1.A and 8.1.B. Let \( f_1 \in \pi_1(CSTF, i(a)) \) and \( f_2 \in \pi_1(STF, i(a)) \) be the classes of oriented (in some way) fibers of the locally trivial \( S^1 \)-fibrations \( pr^1 : CSTF \to STF \) and \( pr^2 : STF \to F \). Then:

1. \( pr^1_*(f_1(\gamma_1)) = f_2^k \in \pi_1(STF, i(a)) \), provided that \( L \) represents an orientation preserving loop on \( F \). (Here the sign of the power of \( f_2 \) depends on the orientation of the fiber we choose to define \( f_2 \).)

2. \( pr^1_*(f_1(\gamma_1)) = 1 \in \pi_1(STF, i(a)) \), provided that \( L \) represents an orientation reversing loop on \( F \). (Here \( 1 \) is a unit element of \( \pi_1(STF, i(a)) \).)

3. \( t_0(\gamma_2) = f_1^k \in \pi_1(CSTF, i(a)) \) for some nonzero \( k \in \mathbb{Z} \).

9.2.E. **Proof of Proposition 9.2.D.** Under the deformation of \( L \) described by the loop \( \gamma_1 \) the point \( L(a) \) never leaves a small neighborhood of its original position.

If \( L \) represents an orientation preserving loop on \( F \), then the coorienting normal of the front at \( L(a) \) is rotated by \( 2\pi \) under \( \gamma_1 \). Thus, the trajectory of \( \alpha \) under the lifting of \( \gamma_1 \) to a loop in the space of Legendrian curves represents a class of an oriented (in some way) fiber of the fibration \( pr^2 \). It is clear that this trajectory coincides with \( pr^1_*(t_0(\gamma_1)) \) and we proved the first statement of the Proposition.

The proof of the second statement of the proposition is similar. The difference is that for \( L \) representing an orientation reversing loop on \( F \) the kink has to slide two times through \( L(a) \) before it comes to its original position. The directions of the rotation of the coorienting normal of the front during these events are opposite. Thus, the total angle of rotation of the coorienting normal is zero. Hence, the trajectory of \( \alpha \) under the lifting of \( \gamma_1 \) to a loop in the space of Legendrian curves is a contractible loop in \( STF \).

To prove the third statement we observe that, if we move an oriented fiber around the loop \( \alpha \subset CSTF \), then in the end it comes to itself, either with the same, or with the opposite orientation. Thus, for any \( \alpha \in \pi_1(CSTF, i(a)) \) either \( f_1 \alpha = \alpha f_1 \), or \( f_1 \alpha = \alpha f_1^{-1} \). Hence, the only elements of \( \pi_1(CSTF, i(a)) \), which are conjugate to \( f_1^k \) are \( f_1^{-k} \) and \( f_1^k \). This means, that to prove the third statement it is sufficient to prove the corresponding fact for any loop, which is free homotopic to \( \gamma_2 \).

Thus, we can assume, that under the loop \( \gamma_2 \) the point \( L(a) \) never leaves a small neighborhood of its original position; and that \( L \) from the very beginning has the property that the cusps of \( L \) are close and the side of \( L \) they are pointing to alternates. (The notion of side is locally well-defined.)
Under the sliding through $L(a)$ of a pair of adjacent cusp points pointing to different (locally well-defined) sides of $L$ the coorienting normal at $L(a)$ is rotated first by half a turn in one direction, then by half a turn in the other direction. Since all the cusp points passing through $L(a)$ can be decomposed into a union of such pairs, we get that the total rotation angle of the coorienting normal is zero. (If $L$ represents an orientation reversing loop on $F$ then the number of cusp points of $L$ is odd, but each of them has to pass through $L(a)$ twice under $\gamma_2$, and this gives the decomposition into the desired pairs.) Thus, the trajectory of $a$ under the lifting of $\gamma_2$ to a loop in the space of Legendrian curves in $STF$ represents $1 \in \pi_1(STF, l(a))$. Hence $t(a) = f^k \in \pi_1(CSTF, l(a))$ for some $k \in \mathbb{Z}$. To prove the statement we have to show, that $k \neq 0$.

One checks that there are only two points in the $S^1$-fiber of $CSTF$ over $l(a)$, which correspond to the front having a cusp at $L(a)$. A Lemma proved by Arnold [4] says, that under the deformations of the wave front shown in Figure 20, the velocity vector at the point $l(a)$ is turning in the direction dependent only, on whether it is true or not, that after the deformation the coorienting normal is pointing to the same direction as the curvature vector. (In the Figure 20 the marked point is frozen into the surface together with the coorienting normal at it.)

![Figure 20](image-url)

It is clear that the loop $\gamma_2$ is homotopic to $\gamma_2'$, in which the point $L(a)$ is frozen into $F$ together with the coorienting normal at it. In the loop $\gamma_2'$ the pair of adjacent cusp points is passing through $L(a)$ in the way shown in Figure 21. One checks (using Arnold's lemma) that under the passage of a pair of cusps
through \( L(a) \) the direction of the velocity vector of \( l \) at \( l(a) \) is rotated by a fiber of \( p^1 \). (Note, that in the construction of \( \gamma_2 \) the front was changed, so that: 1) The (locally well-defined) side of \( L \), the cusps were pointing to, is alternating. 2) All the pairs of adjacent cusps are of the type shown in Figure 21.)

![Figure 21](image_url)

Finally, we note that all the pairs of cusps passing through \( L(a) \) induce rotation of the tangent vector of \( l \) at \( l(a) \) in the same direction. (This is once again because we deformed \( L \), so that the side of \( L \), the cusps are pointing to, alternates.) Thus the total degree of \( f_1 \) in \( t_a(\gamma_2) \) is non zero. This finishes the proof of the Proposition. ⧫

9.2.F. PROPOSITION. Let \( f_1 \in \pi_1(CSTF, \bar{l}(a)) \) and \( f_2 \in \pi_1(STF, l(a)) \) be the classes of oriented (in some way) fibers of the \( S^1 \)-fibrations \( p^1 : CSTF \to STF \) and \( p^2 : STF \to F \), respectively. Then:

1. \( f_1 \alpha = \alpha f_1 \) (in \( \pi_1(CSTF, \bar{l}(a)) \)), for any \( \alpha \in \pi_1(CSTF, \bar{l}(a)) \).

2. \( f_2 \alpha = \alpha f_2 \) (in \( \pi_1(STF, l(a)) \)), for any \( \alpha \in \pi_1(STF, l(a)) \), which projects to an orientation preserving loop on \( F \).

3. \( f_2 \alpha = \alpha f_2^{-1} \) (in \( \pi_1(STF, l(a)) \)), for any \( \alpha \in \pi_1(STF, l(a)) \), which projects to an orientation reversing loop on \( F \).

9.2.G. Proof of Proposition 9.2.F. To prove the first statement we observe that, if we move an oriented fiber around the loop \( \alpha \subset CSTF \) then in the end it comes to itself, with either the same or with the opposite orientation. Thus, for any \( \alpha \in \pi_1(CSTF, \bar{l}(a)) \) either \( f_1 \alpha = \alpha f_1 \), or \( f_1 \alpha = \alpha f_1^{-1} \).

It is clear that \( \alpha \in \pi_1(CSTF, \bar{l}(a)) \) can be realized as a lifting \( \bar{f}_\alpha \) of some front \( L_\alpha \). Consider the deformation of \( L_\alpha \) described by the loop \( \gamma_2 \). We know that \( t_a(\gamma_2) = f_2^k \) for some nonzero \( k \in \mathbb{Z} \) (see 9.2.D), and that \( t_a(\gamma_2) \in Z(\bar{l}) < \pi_1(CSTF, \bar{l}(a)) \) (see 9.2.B). Thus, \( f_2^k \bar{f}_\alpha = \bar{f}_\alpha f_2^{-k} \) and by the above observation we get, that \( f_2 \) commutes with \( \bar{f}_\alpha \). This finishes the proof of the first statement.
We note that a local orientation of the neighborhood of $L(a)$ on $F$ induces an orientation of the fiber of $pr^2$ over $L(a)$. Combining this fact with the observation, that for any $a \in \pi_1(STF, l(a))$ either $f_2(a) = \alpha f_3$, or $f_2(a) = \alpha f_2^{-1}$, we get the proof of the other two statements of the proposition.

9.2.H. Proposition. Let $F \not\in S^2, T^2$ (torus), $\mathbb{R}P^2, K$ (Klein bottle) be a surface (not necessarily compact or orientable) and $G$ be a nontrivial commutative subgroup of $\pi_1(F)$. Then $G$ is infinite cyclic.

9.2.I. Proof of Proposition 9.2.H. It is well known, that any closed $F$, other than $S^2, T^2, \mathbb{R}P^2$, $K$, admits a hyperbolic metric of a constant negative curvature. (It is induced from the universal covering of $F$ by the hyperbolic plane $H$.) The Theorem by A. Pressman (see [6] pp. 258-265) says, that if $M$ is a compact Riemannian manifold with a negative curvature, then any nontrivial Abelian subgroup $G < \pi_1(M)$ is isomorphic to $\mathbb{Z}$. Thus, if $F \not\in S^2, T^2, \mathbb{R}P^2, K$ is closed, then any nontrivial commutative $G < \pi_1(F)$ is infinite cyclic.

If $F$ is not closed then the statement of the Proposition is also true, because in this case $F$ is homotopy equivalent to a bouquet of circles.

9.3. Proof of Lemma 9.0.D. The proof is based on the constructions and the propositions of Section 9.2.

We start by making the following two observations 9.3.A and 9.3.B.

9.3.A. Remark. In $\mathbb{Z}$ there are no elements of finite order. Thus, if $m \neq 0$, then $\Delta_{\mathbb{Z}^r}(q) \neq 0 \iff m \Delta_{\mathbb{Z}^r}(q) = \Delta_{\mathbb{Z}^r}(q^m) \neq 0$. Hence, to prove Lemma 9.0.D it is sufficient to show that $\Delta_{\mathbb{Z}^r}(q^m) = 0$ for a certain power $m \neq 0$ of $q \in \pi_1(L, L)$.

9.3.B. Proposition. Let $q_1, q_2 \in \pi_1(L, L)$ be loops such that:

$$pr^*_1(t_a(q_1)) = pr^*_1(t_a(q_2)) \in \pi_1(STF, l(a))$$

(9.1)

Then $\Delta_{\mathbb{Z}^r}(q_1) = 0$, provided that $\Delta_{\mathbb{Z}^r}(q_2) = 0$.

9.3.C. Proof of the Proposition 9.3.B. The kernel of the homomorphism

$$pr^*_1 : \pi_1(CSTF, l(a)) \to \pi_1(STF, l(a))$$

(9.2)

is generated by $f_1$, the class of an oriented fiber of the fibration $pr_1$. Proposition 9.2.F says, that $f_1$ is in the center of $\pi_1(CSTF, l(a))$. Thus, $t_a(q_2) = t_a(q_1)^j f_1$ for some $j \in \mathbb{Z}$. We know (see 9.2.D) that $t_a(q_2) = f_1^k$ for some nonzero $k \in \mathbb{Z}$. Consider a loop $q_1^k \in \pi_1(CSTF, l(a))$. Thus, $f_1$ is in the center of $\pi_1(CSTF, l(a))$ we get that $t_a(q_2^k) = t_a(q_1)^j t_a(q_2^k)$. Using Proposition 9.2.B we get that $q_2^k = q_1^k q_2$. By the assumption of the Proposition we have $\Delta_{\mathbb{Z}^r}(q_1) = 0$ and $\Delta_{\mathbb{Z}^r}(q_2) = 0$. Using remark 9.3.A we get that $\Delta_{\mathbb{Z}^r}(q_2) = 0$.

We first prove Lemma 9.0.D for $F \not\in S^2, \mathbb{R}P^2, T^2, K$, and then separately for the cases $F = S^2, \mathbb{R}P^2, T^2, K$. 
9.3.D. Case \( F \neq S^2, T^2, \mathbb{R}P^2, K \). Proposition 9.2.B says, that

\[ \pi_1(\mathcal{L}, L) = Z(\tilde{l}) < \pi_1(CSTF, \tilde{I}(a)) \]

The corresponding isomorphism (see Section 9.2.A) maps \( q \in \pi_1(\mathcal{L}, L) \) to \( t_a(q) \in \pi_1(CSTF, \tilde{I}(a)) \) (the trajectory of \( a \) under the lifting of \( q \) by \( \tilde{l} \)).

Thus, for any \( q \in \pi_1(\mathcal{L}, L) \) the elements \( t_a(q) \) and \( \tilde{l} \) commute in \( \pi_1(CSTF) \). Hence, \( L = \text{pr}_2^* \pi_1^{*}(\tilde{l}) \) commutes with \( \text{pr}_2^* \pi_1^{*}(t_a(q)) \) in \( \pi_1(F, L(a)) \). The Proposition 9.2.H implies, that there is a \( Z \) subgroup of \( \pi_1(F, L(a)) \) generated by some \( g \in \pi_1(F, L(a)) \), which contains both of these loops. Then there exist \( m, n \in \mathbb{Z} \), such that \( L = g^m \) and \( \text{pr}_2^* \pi_1^{*}(t_a(q)) = g^n \).

Consider a wave front \( L_1 \) (such that \( l_1(a) = \tilde{I}(a) \)), which represents \( g \). We can lift it to an element \( l_1 \in \pi_1(STF, I(a)) \).

The kernel of \( \text{pr}_2^* \) is generated by \( f_2 \), the class of an oriented fiber, which has infinite order in \( \pi_1(STF) \) for our surfaces \( F \). Using Proposition 9.2.F one can interchange \( f_2 \) with the other elements of \( \pi_1(STF, I(a)) \). We get that \( I = I^m f_2^m \) and \( \pi_1^{*}(t_a(q)) = I^m f_2^{m} \), for some \( i, j \in \mathbb{Z} \).

We prove Lemma 9.0.D separately for the cases \( m \neq 0 \) and \( m = 0 \) in Section 9.3.E and Section 9.3.F, respectively. (Geometrically these two cases correspond to \( L \) representing a homotopically nontrivial, and homotopically trivial loop on \( F \).)

9.3.E. Case \( m \neq 0 \). Remark 9.3.A says that to prove Lemma 9.0.D it is sufficient to show that \( \Delta_{FV}(g^m) \neq 0 \).

We do it by constructing \( \alpha \in \pi_1(\mathcal{L}, L) \) such that \( \text{pr}_1^{*}(t_a(\alpha)) = \text{pr}_1^{*}(t_a(g^m)) \) and \( \Delta_{FV}(\alpha) = 0 \). After this the statement follows from the Proposition 9.3.B.

One can show that:

\[ \text{pr}_1^{*}(t_a(g^m)) = I^m f_2^k \text{ for some } k \in \mathbb{Z}. \]  

(9.3)

For \( g \) which is an orientation preserving loop this follows from the following calculation (which uses 9.2.F):

\[ \text{pr}_1^{*}(t_a(g^m)) = \text{pr}_1^{*}((t_a(g))^m) = (I^m f_2^m = \text{pr}_1^{*}((I^m f_2^m) f_2^{m-ni}) = \text{pr}_1^{*}(I^m f_2^{m-ni}). \]  

(9.4)

(Recall that \( \text{pr}_1^{*}(t_a(q)) = I^m f_2^j \) for some \( j \in \mathbb{Z} \), see section 9.3.D.)

For \( g \), which is an orientation reversing loop on \( F \), this follows from the similar calculation (also based on the proposition 9.2.F).

The fact that \( \text{pr}_1^{*}(t_a(g^m)) \) should commute with \( I \) (since \( t_a(g^m) = (t_a(g))^m \in Z(\tilde{l}) \)) and the Proposition 9.2.F imply, that \( k \) in (9.3) is zero, provided that \( L \) represents an orientation reversing loop on \( F \).

Consider the case of \( L \) representing an orientation preserving loop on \( F \). Let \( \gamma_l \) be the sliding of a kink along \( L \) (see 8.1.A). Proposition 9.2.D says that \( t_a(\gamma_l) = f_2 \). (For a proper choice of the orientation of the fiber used to define \( f_2 \).) Hence, the loop \( \alpha \in \pi_1(\mathcal{L}, L) \), for which \( \text{pr}_1^{*}(t_a(\alpha)) = \text{pr}_1^{*}(t_a(g^m)) \) is: \( n \) times sliding of \( L \) along itself (induced by a rotation of the parameterizing circle) composed with \( \gamma_l \).
II. INVARIANTS OF WAVE FRONTS ON SURFACES

If $L$ represents an orientation reversing loop, then, as we have shown above, $pr_1^*(t_a(q^m)) = 1$. Hence, the loop $\alpha \in \pi_1(L, L)$, for which pr_1^*(t_a(\alpha)) = pr_1^*(t_a(q^m)) is: $n$ times sliding of $L$ along itself (induced by a rotation of the parameterising circle).

No triple point stratum crossings happen under the sliding of $L$ along itself and $\Delta_{ST}(\gamma_1) = 0$ by the assumption of the the Lemma. Hence $\Delta_{ST}(\alpha) = 0$. Proposition 9.3.B implies that $\Delta_{ST}(q^m) = 0$. Thus, we have proved (see 9.3.A) Lemma 9.0.D for $F = S^2, RP^2, T^2, K$ and $m \neq 0$.

9.3.F. Case $m = 0$. If $m = 0$, then $L$ represents $1 \in \pi_1(F, L(a))$.

We want to construct $\alpha \in \pi_1(L, L)$, such that $pr_1^*(t_a(\alpha)) = pr_1^*(t_a(q^2))$ and $\Delta_{ST}(\alpha) = 0$. After this the statement follows from the Proposition 9.3.B and remark 9.3.A.

For any $q \in \pi_1(L, L)$ the projection $pr_2^* pr_1^*(t_a(q^2))$ is an orientation preserving loop on $F$. A straightforward check shows that $\alpha$ can be obtained by a composition of $\gamma_1^{-1}$ (see 8.1.A) and loops obtained by the following construction.

Push $L$ into a small disc by a generic regular homotopy $r$. Slide this small disc along some orientation preserving curve in $F$ and return $L$ to its original shape along $r^{-1}$.

Clearly, the inputs of $r$ and $r^{-1}$ into $\Delta_{ST}$ cancel out and no discriminant crossings happen when we slide the small disc, with the front inside it, along a loop in $F$. By the assumption of the Lemma $\Delta_{ST}(\gamma_1) = 0$. Thus, $\Delta_{ST}(\alpha) = 0$. Proposition 9.3.B implies that $\Delta_{ST}(q^2) = 0$, and we have proved (see 9.3.A) Lemma 9.0.D for $F = S^2, RP^2, T^2, K$.

9.3.G. Case $F = S^2$. One checks that $\pi_1(STS^2) = \mathbb{Z}_2$. Thus, for any $q \in \pi_1(L, L)$ we have $pr_1^*(t_a(q^2)) = 1 = pr_1^*(t_a(1)) \in \pi_1(ST, F, I(a))$. (Here $1$ is a trivial loop in $L$.) Proposition 9.3.B implies that $\Delta_{ST}(q^2) = 0$. This finishes (see 9.3.A) the proof of Lemma 9.0.D for $F = S^2$.

9.3.H. Case $F = T^2$. Using Proposition 9.2.F we get that $\pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. As before we fix $q \in \pi_1(L, L)$ and construct $\alpha \in \pi_1(L, L)$ such that $pr_1^*(t_a(\alpha)) = pr_1^*(t_a(q^2))$.

A straightforward check shows that such $\alpha$ can be constructed as a multiplication of the following elements:

1) The loop $\gamma_1$, which is the sliding of a kink along $L$ (see 8.1.A).
2) The loops $\gamma_2$ and $\gamma_4$, which are slidings of $L$ along the unit vector fields parallel to the meridian and longitude of $T^2$, respectively.

Since $\Delta_{ST}(\gamma_1) = 0$ by the assumption of the Lemma and no discriminant crossings occur under $\gamma_2$ and $\gamma_4$, we get that $\Delta_{ST}(\alpha) = 0$. Proposition 9.3.B implies that $\Delta_{ST}(q^2) = 0$. This finishes the proof of Lemma 9.0.D for $F = T^2$.

9.3.I. Case $F = RP^2$. One checks that $\pi_1(STRP^2) = \mathbb{Z}_4$. Thus $pr_1^*(t_a(q^m)) = 1 = pr_1^*(t_a(1)) \in \pi_1(ST, F, I(a))$ for any $q \in \pi_1(L, L)$. (Here $1$ is a trivial loop in $L$.) Proposition 9.3.B implies that $\Delta_{ST}(q^2) = 0$. This finishes (see 9.3.A) the proof of Lemma 9.0.D for $F = RP^2$. 

9.3. J. Case $F = K$. Proposition 9.2.B says that $\pi_1(C, L)$ is isomorphic to $Z(\bar{I})$. The kernel of the homomorphism

$$\text{pr}_1^*: \pi_1(CSTK, \bar{I}(a)) \to \pi_1(STK, l(a))$$ (9.5)

is generated by $f_1$ (the class of the fiber of $\text{pr}_1$). Proposition 9.2.F says that $f_1$ belongs to the center of $\pi_1(CSTF, \bar{I}(a))$. Thus $\text{pr}_1^*(Z(\bar{I})) = Z(l)$, the centralizer of $l \in \pi_1(STF, l(a))$. We want to show that a certain power of any element of $Z(l)$ can be represented as $\text{pr}_1^*(t_\alpha(a))$ for some $\alpha \in \pi_1(C, L)$, such that $\Delta_{STl}(\alpha) = 0$. This would imply the statement of the Lemma for $F = K$.

Consider $K$ as a quotient of a rectangle modulo the identification on its sides shown in Figure 6. We can assume, that $L(a)$ coincides with the image of a corner of the rectangle. Let $L_1$ and $L_2$ be the wave fronts such that $\bar{I}(a) = \bar{I}_1(a) = \bar{I}_2(a)$, $L_1 = c \in \pi_1(K, L(a))$ and $L_2 = d \in \pi_1(K, l(a))$. (Here $c$ and $d$ are the elements of $\pi_1(K)$ realized by the images of the sides of the rectangle used to construct $K$, see Figure 6.) One can show that:

$$\pi_1(STK, l(a)) = \{i_1, i_2, j_1, j_2 | i_1^{f_1} i_2^{f_2} j_1^{f_1} j_2^{f_2} = i_1^{f_1} j_2^{f_2} = j_1^{f_1} i_2^{f_2} = j_2^{f_1} i_1^{f_2}, i_1 j_2 = j_2 i_1\}.$$ (9.6)

The second and the third relations in this presentation follow from the Proposition 9.2.F. To get the first relation one notes that the identity $dc^{k} = c^{f_1} d \in \pi_1(K, L(a))$ implies $i_2^{f_1} i_1^{f_2} = i_1^{f_1} j_2^{f_2}$ for some $k \in \mathbb{Z}$. But $j_2$ commutes with $i_1$, since they can be lifted to $STT^2$, the fundamental group of which is Abelian. Hence, $k = 0$.

Using relations (9.6) one can calculate $Z(l)$. (Note, that these relations allow one to present any element of $\pi_1(STK, l(a))$ as $i_1^{f_1} i_2^{f_2} j_1^{f_1} j_2^{f_2}$, for some $i, j, k \in \mathbb{Z}$.)

A straightforward (but long) check, which uses (9.6), shows that:

a) If $L$ represents an orientation preserving loop on $K$, then a certain power of any element of $Z(l)$ can be obtained as $\text{pr}_1^*(t_\alpha(a))$, for $\alpha$ being a product of $\gamma_1$ (see 8.1.A), $\gamma_3, \gamma_4$, described below, and their inverses.

b) If $L$ represents an orientation reversing loop on $K$, then a certain power of any element of $Z(l)$ can be obtained as $\text{pr}_1^*(t_\alpha(a))$, for $\alpha$ being a product of $\gamma_4, \gamma_6$, described below, and their inverses.

To get $\gamma_7$ consider an orientation covering $p : T^2 \to K$. There is a loop $\beta$ in the space of all autodiffeomorphisms of $T^2$, which is a sliding of $T^2$ along the unit vector field parallel to the lifting of $c$ (see Figure 6). We can assume, that $\beta$ agrees with the covering structure of $p$. The loop $\gamma_3$ is the projection of the sliding of $p^{-1}(L)$, induced by $\beta$.

Consider a loop $\beta$ in the space of all the autodiffeomorphisms of $K$, which is a sliding of $K$ along the unit vector field parallel to the curve $d$ on $K$. (Note that $K$ has to slide twice along itself under this loop, before every point of it comes to the original position.) The loop $\gamma_4$ is the sliding of $L$ induced by $\beta$.

The loop $\gamma_6$ is a sliding of $L$ along itself, induced by a rotation of the parameterizing circle.

No discriminant crossings occur under $\gamma_3, \gamma_4$ and $\gamma_6$. (For the loop $\gamma_6$ this statement is a little bit less trivial.) By the assumption of the Lemma $\Delta_{STl}(\gamma_1) = 0$. Thus, $\Delta_{STl}(\alpha) = 0$ and because of the reasons explained in the beginning of section 9.3.J we got the proof of the Lemma for $F = K$. 


This finishes the proof of Lemma 9.0.D. \( \square \)

10. Proof of Theorem 7.2.A

We denote by \( \Delta_{\mathcal{F}}(\gamma_1) \) and \( \Delta_{\mathcal{F}}(\gamma_2) \) the change (see 8.1) of \( \mathcal{F} \) along the loops \( \gamma_1 \) and \( \gamma_2 \) (see 8.1.A and 8.1.B). Theorem 8.2.A implies that to prove the Theorem 7.2.A it is enough to show that \( \Delta_{\mathcal{F}}(\gamma_1) = 0 \) and \( \Delta_{\mathcal{F}}(\gamma_2) = 0 \).

Let us show that \( \Delta_{\mathcal{F}}(\gamma_1) = 0 \). We start by noticing that \( \Delta_{\mathcal{F}} \) under the deformation \( r \) of \( L \) to a front with two opposite kinks, cancels out with \( \Delta_{\mathcal{F}} \) under the deformation of \( L \) along \( r^{-1} \). Thus, we have to show that \( \Delta_{\mathcal{F}} \) under the sliding of the kink along \( L \) is zero. Note that the only crossings of a dangerous selftangency stratum, which occur under this sliding, happen either when the kink passes through a double point, or when it passes through a cusp point of \( L \).

The kink passes twice through each double point of \( L \). (Once along each intersecting branch.) As one can check (see Figure 22) the two \( T \)-equivalence classes corresponding to these events are equal and the signs of the corresponding dangerous selftangency stratum crossings are opposite. Thus, the corresponding two terms in \( \Delta_{\mathcal{F}} \) cancel out. (One can check that this part of the proof would not go through, if \( F \) is nonorientable and the double point separates the front into two orientation reversing loops.)

![Figure 22](image.png)

One can see (using Figure 15) that either two or zero crossings of the dangerous selftangency stratum occur under the passage of the kink through a cusp. If the number of crossings is zero, then clearly there is no input into \( \Delta_{\mathcal{F}} \). In the case of two crossings one checks that the signs of them are opposite and the
corresponding $T$-equivalence classes are equal. Thus, the corresponding two terms of $\Delta_{TT}$ also cancel out and we proved that $\Delta_{TT}(\gamma_1) = 0$.

We are left to prove that $\Delta_{TT}(\gamma_2) = 0$. Consider the case of $L$ having a nonzero Maslov index. One can see that the input into $\Delta_{TT}$ of the deformation $r$ of $L$ to a front with cusps pointing to alternating (locally well-defined) sides of $L$ cancels out with the input into $\Delta_{TT}$ of the deformation along $r^{-1}$. All the dangerous selftangency stratum crossings, which are left, occur when a cusp passes through a double point of the front. We decompose the set of cusps passing through a double point into the pairs of adjacent cusps. One can check (see Figure 23) that the inputs into $\Delta_{TT}$ of the cusps from the same pair cancel out. And we proved that $\Delta_{TT}(\gamma_2) = 0$ in the case of $L$ having a nonzero Maslov index.

The proof of $\Delta_{TT}(\gamma_2) = 0$ in the case of $L$, having a zero Maslov index is similar. But one has to take care about some extra crossings of the dangerous selftangency stratum, which happen when we are artificially creating (and later killing) the four cusps, two of which slide around $L$. (See Figure 17.)

This finishes the Proof of Theorem 7.2.A. □
III. Homotopy Groups of the Space of Curves on a Surface

11. Main Results

11.1. Basic definitions. A curve is a smooth immersion of (an oriented circle) $S^1$ into a (smooth) surface $F$. For a surface $F$ we denote by $F$ the space of all curves on $F$.

Two curves $s_0$ and $s_1$ are said to be regularly homotopic, if there exists a homotopy $H : S^1 \times I \to F$, such that $H(t \times 0) = s_0(t)$, $H(t \times 1) = s_1(t)$ and $H(\bullet \times x)$ is an immersion for every $x \in I$. This means that $s_0$ and $s_1$ are in the same connected component of $F$.

Two (oriented) curves with a tangency point, at which the velocity vectors of the two curves are pointing in the same direction are said to be direct tangent to each other at this point.

For a surface $F$ we denote by $STF$ the spherical tangent bundle of $F$ and by $pr : STF \to F$, the corresponding locally trivial $S^1$-fibration.

For a curve $\xi$ on $F$ we denote by $\tilde{\xi}$ its lifting to $STF$, which maps every point $t \in S^1$ to the direction of the velocity vector of $\xi$ at $t$.

We fix a point $a$ on $S^1$. Then a curve $\xi$ represents an element of $\pi_1(F, \xi(a))$ and $\tilde{\xi}$ represents an element of $\pi_1(STF, \tilde{\xi}(a))$. When there is no ambiguity, we denote these two elements by $\xi$ and $\tilde{\xi}$, respectively.

11.2. Fundamental group of the space of curves on an orientable surface. For orientable surfaces the group $\pi_1(F, \xi)$ appears to be much simpler than for nonorientable surfaces.

11.2.A. Theorem. Let $F = S^2$ and let $\xi$ be a curve on $S^2$. Then $\pi_1(F, \xi) = \mathbb{Z}$.

11.2.B. Theorem. Let $F = T^2$ (torus) and let $\xi$ be a curve on $T^2$. Then $\pi_1(F, \xi) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

11.2.C. Theorem. Let $F \neq S^2, T^2$ be an orientable surface (not necessarily compact) and let $\xi$ be a curve on $F$.

I. If $\xi$ represents a homotopically nontrivial loop on $F$, then $\pi_1(F, \xi) = \mathbb{Z} \oplus \mathbb{Z}$.

II. If $\xi$ represents a homotopically trivial loop on $F$, then $\pi_1(F, \xi) = \pi_1(STF)$. 

51

11.3. Fundamental group of the space of curves on a nonorientable surface.

11.3.A. Theorem. Let $F = \mathbb{R}P^2$ and $\xi$ be a curve on $\mathbb{R}P^2$. Then $\pi_1(F, \xi) = \mathbb{Z}_4$.

11.3.B. Theorem. Let $F = K$ (Klein bottle) and let $\xi$ be a curve on $K$.

I. If $\xi$ realizes an orientation preserving loop on $K$, then $\pi_1(F, \xi) = \pi_1(STK)$, provided that $\bar{\xi} = b\xi$ in $\pi_1(STK, \bar{\xi}(a))$ for some $b \in \pi_1(STK, \bar{\xi}(a))$, which projects to an orientation reversing loop on $K$, and $\pi_1(F, \xi) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, otherwise.

II. If $\xi$ represents an orientation reversing loop on $K$, then $\pi_1(F, \xi)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The following construction will be needed for a description of $\pi_1(F, \xi)$ for $\xi$ representing a homotopically nontrivial loop on $F$ and $F \neq \mathbb{R}P^2, K$.

Let $F \neq \mathbb{R}P^2$, $K$ be a surface (not necessarily compact) and let $\xi$ be a curve on $F$, such that $\xi \neq 1 \in \pi_1(F, \xi(a))$. Let $f \in \pi_1(STF, \bar{\xi}(a))$ be the homotopy class of an oriented fiber of the $S^1$-fibration $pr : STF \to F$.

One can show, that there exists a unique maximal Abelian subgroup $G_\xi < \pi_1(F, \xi(a))$ containing $\xi \in \pi_1(F, \xi(a))$ and that this $G_\xi$ is isomorphic to $\mathbb{Z}$ (see also Proposition 12.1.H). Let $g$ be its generator. Consider a curve $g_\xi$ direct tangent to $\xi$ at $\xi(a)$, which realizes $g \in \pi_1(F, \xi(a))$.

One can show, that $\bar{\xi} \in \pi_1(STF, \bar{\xi}(a))$ can be presented in the unique way as $\bar{g}^k f^l \in \pi_1(STF, \bar{\xi}(a))$ (see also the Proof of Theorem 11.3.C).

11.3.C. Theorem. Let $F \neq \mathbb{R}P^2, K$ be a nonorientable surface (not necessarily compact) and $\xi$ be a curve on $F$.

I. If $\xi$ represents an orientation reversing loop on $F$, then $\pi_1(F, \xi) = \mathbb{Z}$.

II. If $\xi$ represents a homotopically nontrivial orientation preserving loop on $F$ then:

a) $\pi_1(F, \xi) = \mathbb{Z} \oplus \mathbb{Z}$, provided that $g_\xi$ is an orientation preserving loop on $F$, or that $g_\xi$ is an orientation reversing loop and $\bar{\xi} = (\bar{g}_\xi)^{2k} f^l$ for some nonzero $k$ and $l$.

b) $\pi_1(F, \xi) = \pi_1(K)$, provided that $\bar{\xi} = (\bar{g}_\xi)^{2k}$ for some nonzero $k$ and that $g_\xi$ is an orientation reversing loop.

III. If $\xi$ represents a homotopically trivial loop, then:

a) $\pi_1(F, \xi)$ is isomorphic to the subgroup of $\pi_1(STF)$ consisting of all the elements, which project to orientation preserving loops on $F$, provided that $\bar{\xi}$ is a homotopically nontrivial loop in $STF$. (This means, cf. 12.1.B, that $\xi$ is not regularly homotopic to the figure eight curve.)

b) $\pi_1(F, \xi) = \pi_1(STF)$, provided that $\xi$ is a homotopically trivial loop in $STF$. (This means, cf. 12.1.B, that $\xi$ is regularly homotopic to the figure eight curve.)
III. HOMOTOPY GROUPS OF THE SPACE OF CURVES ON A SURFACE

Theorems 11.3.A, 11.3.B and 11.3.C are proved in Sections 12.4, 12.5 and 12.6, respectively.

11.4. Higher homotopy groups of the space of curves.

11.4.A. Theorem. Let $F$ be a surface (not necessarily compact or orientable) and $\xi$ be a curve on $F$.

I. If $F$ is equal to $S^2$ or $\mathbb{R}P^2$, then $\pi_2(F, \xi) = \mathbb{Z}$ and $\pi_n(F, \xi) = \pi_n(S^2) \oplus \pi_{n+1}(S^2)$, $n \geq 3$.

II. If $F \neq S^2, \mathbb{R}P^2$, then $\pi_n(F, \xi) = 0$, $n \geq 2$.

For the Proof of Theorem 11.4.A see Section 12.7.

12. Proofs

12.1. Some useful facts and technical Lemmas.

12.1.A. Lemma. Let $F$ be a surface, $STF$ be its spherical tangent bundle and $p \in STF$ be a point. Let $f \in \pi_1(STF, p)$ be the class of an oriented (in some way) fiber of the $S^1$-fibration $pr : STF \to F$.

If $\alpha \in \pi_1(STF, p)$ is a loop, which projects to an orientation preserving loop on $F$, then

$$\alpha f = f \alpha.$$ (12.1)

If $\alpha \in \pi_1(STF, p)$ is a loop, which projects to an orientation reversing loop on $F$, then

$$\alpha f = f^{-1} \alpha.$$ (12.2)

The proof of this Lemma is straightforward.

12.1.B. Parametric h-principle. The parametric h-principle, see [7] page 16, implies that $F$ is weak homotopy equivalent to the space $\Omega STF$ of free loops in $STF$. The corresponding mapping $h : F \to \Omega STF$ sends an immersion $\xi \in F$ to a loop $\tilde{\xi} \in \Omega STF$ by mapping a point $y \in S^1$ to the point in $STF$, corresponding to the velocity vector of $\xi$ at $y$.

12.1.C. Relations between the homotopy groups of the space $STF$ and of the space of free loops in $STF$. Let $b$ be a point on $STF$. We denote by $\Omega_b STF$ the space of all loops (in $STF$) based at $b$.

Let $\Omega STF$ be the space of all the free loops in $STF$ and $\lambda$ be a fixed element of $\Omega STF$. Fix a point $a$ on $S^1$.

Let $t : \Omega STF \to STF$ be the mapping, which sends $\omega \in \Omega STF$ to $\omega(a) \in STF$. One checks, that this $t$ is a Serre fibration with the fiber of it isomorphic to the space of loops based at the corresponding point.

This fibration gives rise to the following long exact sequence:

$$\cdots \longrightarrow \pi_n(\Omega \lambda(a) STF, \lambda) \overset{\text{inc}}{\longrightarrow} \pi_n(\Omega STF, \lambda) \overset{t_*}{\longrightarrow} \pi_n(STF, \lambda(a)) \overset{\partial}{\longrightarrow} \cdots.$$ (12.3)

The following fact is well-known.
12.1.D. Let \( \lambda \) be a loop (not necessarily contractible) on \( STF \) and \( a \) be a fixed point on \( S^1 \), then for any \( n \geq 0 \):

\[
\pi_n(\Omega \lambda(a) STF, \lambda) = \pi_{n+1}(STF, \lambda(a)).
\]

12.1.E. Proposition. The group \( \pi_1(\Omega STF, \lambda) \) is isomorphic to \( Z(\lambda) \), the centralizer of an element \( \lambda \in \pi_1(STF, \lambda(a)) \).

12.1.F. Proof of Proposition 12.1.E. Let \( p : \Omega STF \to STF \) be the mapping, which sends \( \omega \in \Omega STF \) to \( \omega(a) \in STF \). (One can check, that this \( p \) is a Serre fibration, with the fiber of it isomorphic to the space of loops based at the corresponding point.)

A Proposition proved by V.I. Hansen [8] says that: if \( X \) is a topological space with \( \pi_2(X) = 0 \), then \( \pi_1(\Omega X, \omega) = \tilde{Z}(\omega) < \pi_1(X, \omega(a)) \).

(Here \( \Omega X \) is the space of free loops in \( X \) and \( \omega \) is an element of \( \Omega X \).) One can check that \( \pi_2(STF) = 0 \) for any surface \( F \). Thus, we get that \( \pi_1(\Omega STF, \lambda) \) is isomorphic to \( Z(\lambda) < \pi_1(STF, \lambda(a)) \). From the proof of the Hansen's Proposition it follows that the isomorphism is induced by \( p_* \). \( \square \)

The following statement is an immediate consequence of Lemma 12.1.E and the \( h \)-principle.

12.1.G. Corollary. Let \( F \) be a surface and \( \xi \) be a curve on \( F \), then \( \pi_1(F, \xi) \) is isomorphic to \( Z(\xi) \), the centralizer of \( \xi \in \pi_1(STF, \xi(a)) \).

12.1.H. Lemma. Let \( F \neq S^2, T^2 \) (torus), \( \mathbb{R}P^2, K \) (Klein bottle) be a surface (not necessarily compact or orientable) and let \( G' \) be a nontrivial commutative subgroup of \( \pi_1(F) \). Then \( G' \) is infinite cyclic and there exists a unique maximal subgroup \( G < \pi_1(F) \), such that \( G' < G \) and that \( G \) is infinite cyclic.

12.1.I. Proof of Lemma 12.1.H. It is well known that any closed \( F \), other than \( S^2, T^2, \mathbb{R}P^2, K \), admits a hyperbolic metric of a constant negative curvature, which is induced from the universal covering of it by the hyperbolic plane \( H \). The Theorem by A. Preissman (see [6] pp. 258-265) says, that if \( M \) is a compact Riemannian manifold with a negative curvature, then any nontrivial Abelian subgroup \( G' < \pi_1(M) \) is isomorphic to \( Z \). Thus, if \( F \neq S^2, T^2, \mathbb{R}P^2, K \) is closed, then any nontrivial commutative \( G' < \pi_1(F) \) is infinite cyclic.

The proof of the Preissman's Theorem given in [6] is based on the fact, that if \( \alpha, \beta \in \pi_1(M) \) are nontrivial commuting elements, then there exists a geodesic in \( \tilde{M} \) (the universal covering of \( M \)), which is mapped to itself under the action of these elements considered as deck transformations on \( \tilde{M} \). Moreover, these transformations, restricted to the geodesic, act as translations. This implies, that if \( F \neq S^2, T^2, \mathbb{R}P^2, K \) is a closed surface, then there exists a unique maximal infinite cyclic \( G < \pi_1(F) \) such that \( G' < G \). This gives the proof of the Lemma for closed \( F \).

If \( F \) is not closed, then this statement is also true, because in this case \( F \) is homotopy equivalent to a bouquet of circles. \( \square \)

12.2. Proof of Theorem 11.2.A. From the exact homotopy sequence of the fibration \( pr : STS^2 \to S^2 \) we get, that \( \pi_1(S^2) = \mathbb{Z}_2 \). Lemma 12.1.G implies, that \( \pi_1(F, \xi) = \mathbb{Z}_2 \). \( \square \)
12.3. Proof of Theorem 11.2.2. From the exact homotopy sequence of the
fibration $pr : STT^2 \to T^2$ and identity (12.1) we get, that $\pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Lemma 12.1.G implies, that $\pi_1(F, \xi) = \pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. □

12.4. Proof of Theorem 11.3.A. From the exact homotopy sequence of
the fibration $pr : STRP^2 \to RP^2$ we get, that $\pi_1(STRP^2) = \mathbb{Z}_4$. Lemma 12.1.G
implies, that $\pi_1(F, \xi) = \mathbb{Z}_4$. □

12.5. Proof of Theorem 11.3.B. Lemma 12.1.G says, that $\pi_1(F, \xi) =
Z(\xi) < \pi_1(STK, \xi(a))$.

Consider $K$ as a quotient of a rectangle modulo the identification on its sides,
which is shown in Figure 6. We can assume, that $\xi(a)$ coincides with the image
of a corner of the rectangle, and $\xi$ is direct tangent to the curve $c$ at $\xi(a)$. Let
g and $h$ be the curves such that: $\xi(a) = \bar{g}(a) = \bar{h}(a), g = c \in \pi_1(K, L(a))$
and $h = d \in \pi_1(K, L(a))$. (Here $c$ and $d$ are the elements of $\pi_1(K)$ realized by the
images of the sides of the rectangle used to construct $K$, see Figure 6.) Let $f$
be the class of an oriented fiber of the fibration $pr : STK \to K$. One can show that:

$$\pi_1(STK, \xi(a)) = \{g, \bar{h}, f | \bar{g}^{-1} \bar{h}, \bar{h} f^{-1} = f \bar{f} \bar{h}, \bar{g} f = f \bar{g}\}. \quad (12.4)$$

The second and the third relations in this presentation follow from (12.1)
and (12.2). To get the first relation one notes that the identity $dc^{k,l} = c^{k,l}$
implies that $\bar{g}^{-1} \bar{h} \bar{f}^{-1} = \bar{g}^{-1} \bar{h} \bar{f}^{-1}$ for some $k \in \mathbb{Z}$. But $\bar{h}^2$
commutes with $\bar{g}$, since they can be lifted to $STT^2$, the fundamental group of which is Abelian.
Hence, $k = 0$.

Using relations (5.4) one can calculate $Z(\xi) = \pi_1(F, \xi)$. (Note that these
relations allow one to present any element of $\pi_1(STK, \xi(a))$ as $\bar{g}^k \bar{h}^l f^m$, for some
$k, l, m \in \mathbb{Z}$.)

This group appears to be:
a) The whole group $\pi_1(STK, \xi(a))$, provided that $\xi = \bar{h}^2 l$ for some $l \in \mathbb{Z}$.
b) An isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(STK, \xi(a))$, provided that
$\xi = \bar{g}^k \bar{h}^2 f^m$ for some $k, l, m \in \mathbb{Z}$, such that either $k \neq 0$ or $m \neq 0$. This
subgroup is generated by $\{ \bar{g}, \bar{h}^2, f \}$.
c) An isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(STK, \xi(a))$, provided that $\xi = \bar{g}^k \bar{h}^{2m+1} f^m$ for some $k, l, m \in \mathbb{Z}$. This subgroup is generated by $\{ \xi, \bar{h}^2 \}$.

Now the statement of the Theorem follows from relations (12.4). □

12.6. Proof of Theorem 11.2.2 and Theorem 11.3.C. We are going to
prove, that the statement of Theorem 11.3.C is true for any orientable surface
$F \neq S^2, T^2$ and any non-orientable $F \neq RP^2, K$.

Clearly this gives a proof of Theorem 11.3.C. Theorem 11.2.C is also an
immediate consequence of this fact.

12.6.A. Proof of Theorem 11.2.2 and Theorem 11.3.C in the case of
$\xi \neq 1 \in \pi_1(F, \xi(a))$. Consider a subgroup $G'$ of $\pi_1(F, \xi(a))$ generated by $\xi$. It
is an infinite cyclic group (see 12.1.H). There is a unique (see 12.1.H) maximal
subgroup $G < \pi_1(F, \xi(a))$, such that $G' < G$ and $G$ is isomorphic to $\mathbb{Z}$. Let $g$ be
the generator of $G$. Let $g_\xi$ be a curve direct tangent to $\xi$ at $\xi(\alpha)$, representing this $g$.

Corollary 12.1.G says that $\pi_1(F, \xi)$ is isomorphic to $Z(\xi)$. Take $\alpha \in Z(\xi)$. Since $\xi$ and $\xi$ commute in $\pi_1(STF, \xi(\alpha))$ we get, that their images under the projection $pr_*: \pi_1(STF, \xi(\alpha)) \to \pi_1(F, \xi(\alpha))$ commute in $\pi_1(F, \xi(\alpha))$. Lemma 12.1.H implies, that these projections are in the subgroup $G$.

The kernel of the homomorphism $pr_*$ is generated by $f$, the homotopy class of an oriented fiber of the $S^1$-fibration $pr: STF \to F$. This fact and identities (12.1) and (12.2) show, that there exist unique $k, l, m, n \in \mathbb{Z}$, such that $\xi = g_\xi f^k$ and $\alpha = g_\alpha f^m$.

Using identities (12.1) and (12.2) we can check for which values of $k, l, m, n$ the elements $\alpha$ and $\xi$ commute. This allows us to calculate $Z(\xi)$. It turns out to be:

a) A group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\{g_\xi, f\}$, provided that $g$ is an orientation preserving loop on $F$.

b) A group isomorphic to $\mathbb{Z}$ generated by $g_\xi f^k$, provided that $g_\xi$ is an orientation reversing loop, and $k$ is odd. (This means, that $\xi$ represents an orientation reversing loop on $F$.) Note also, that in this case $(g_\xi f^k)^2 = g_\xi^2$.

c) A group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\{g_\xi^2, f\}$, provided that $g_\xi$ is an orientation reversing loop on $F$, $k \neq 0$ is even and $l \neq 0$.

d) A group isomorphic to $\pi_1(K)$ generated by $\{g_\xi f\}$, provided that $g_\xi$ is an orientation reversing loop on $F$, $k \neq 0$ is even and $l = 0$.

(Note that if $k = 0$ then $\xi = 1 \in \pi_1(STF, \xi(\alpha))$, which contradicts to our assumption.)

This finishes the proof of the two theorems for this case.

12.6.B. Proof of Theorem 11.2.C and Theorem 11.3.C in the case of $\xi = 1 \in \pi_1(F, \xi(\alpha))$. From the exact homotopy sequence of the $S^1$-fibration $pr: STF \to F$ we get, that $ker pr_*$ is generated by $f$, the homotopy class of the fiber. Since $\xi = 1 \in \pi_1(F, \xi(\alpha))$ we get that there exists $k \in \mathbb{Z}$, such that $\xi = f^k$. Lemma 12.7.A says that $\pi_1(F, \xi)$ is isomorphic to $Z(\xi) = Z(f^k) < \pi_1(STF, \xi(\alpha))$.

For $k \neq 0$ identities (12.1) and (12.2) imply, that $Z(f^k)$ coincides with the set of elements of $\pi_1(STF, \xi(\alpha))$, which project to orientation preserving loops on $F$. This finishes the proof of the Theorem for $\xi = 1 \in \pi_1(F, \xi(\alpha))$ and $\xi \neq 1 \in \pi_1(STF, \xi(\alpha))$.

If $k = 0$, then $\xi = 1 \in \pi_1(STF, \xi(\alpha))$. Thus $Z(\xi) = \pi_1(STF, \xi(\alpha))$. Hence, in this case $\pi_1(F, \xi) = \pi_1(STF, \xi(\alpha))$.

12.7. Proof of Theorem 11.4.A. The proof of this Theorem is based on the following exact sequence, which was introduced in section 12.1.C.

\[
\cdots \to \pi_n(\Omega_{\lambda(\alpha)}STF, \lambda) \xrightarrow{\iota} \pi_n(\Omega STF, \lambda) \xrightarrow{\iota} \pi_n(STF, \lambda(\alpha)) \xrightarrow{\iota} \cdots \quad (12.5)
\]
12.7.A. **Lemma.** If $F$ is equal to $S^2$ or $\mathbb{RP}^2$ and $n \geq 2$, then

$$
\pi_n(\Omega STF, \lambda) = \pi_n(\Omega_{\lambda(a)} STF, \lambda) \oplus \pi_n(STF, \lambda(a)).
$$

12.7.B. **Proof of Lemma 12.7.A.** Fix $n > 1$. We construct a homomorphism $g : \pi_n(STF, \lambda(a)) \to \pi_n(\Omega STF, \lambda)$, such that $t_* \circ g = \text{id}_{\pi_n(STF, \lambda(a))}$. After this, the exactness of the sequence (12.5) implies the statement of the Lemma.

We describe this construction for $F = \mathbb{RP}^2$. The construction of $g$ for $F = S^2$ can be easily deduced from this one.

From the exact homotopy sequence of the covering $STS^2 \to ST\mathbb{RP}^2$ we get, that $\pi_n(ST\mathbb{RP}^2), n \geq 2$, is canonically isomorphic to $\pi_n(STS^2)$.

Take $s : S^n \to ST\mathbb{RP}^2$, which represents a given element of $\pi_n(ST\mathbb{RP}^2, \lambda(a))$. Let $s' : S^n \to STS^2$ be the mapping, which is a lifting of $s$, under the covering $STS^2 \to ST\mathbb{RP}^2$. Fix an orientation on $S^2$. Then for every $x \in S^n$ the orientation on a small neighborhood of $pr s'(x) \in S^2$ induces an orientation on a small neighborhood of $pr s(x) \in \mathbb{RP}^2$.

There is a unique isometric autodiffeomorphism $I_x$ of $\mathbb{RP}^2$, such that:

a) It maps $pr s(*)$ to $pr s(x)$.

b) The differential of it sends $s(*)$ to $s(x)$.

c) The above described local orientation at $pr s(x)$, coincides with the one induced by the differential of $I_x$ from the local orientation at $pr s(*)$.

Let $\hat{s} : S^n \to \Omega ST\mathbb{RP}^2$ be a mapping, which maps $x \in S^n$ to $I_x(\lambda)$ (the translation of $\lambda$ by $I_x$).

Set the value of $g$ on the element of $\pi_n(ST\mathbb{RP}^2, \lambda(a))$, represented by $s$, to be the element of $\pi_n(\Omega ST\mathbb{RP}^2, \lambda)$ represented by $\hat{s}$. One checks that this $g$ is the desired homomorphism from $\pi_n(ST\mathbb{RP}^2, \lambda(a))$ to $\pi_n(\Omega ST\mathbb{RP}^2, \lambda)$.

This finishes the proof of Lemma 12.7.A. □

12.7.C. One checks, that $\pi_2(STF) = 0$ and $\pi_n(STF) = \pi_n(S^2), n \geq 3$, for $F$ equal to $S^2$ or $\mathbb{RP}^2$. Now Lemma 12.7.A, Lemma 12.1.D and the weak homotopy equivalence given by the $h$-principle (see 12.1.B) imply the first statement of the Theorem. (Note, that $\pi_3(S^2) = \mathbb{Z}$.)

Lemma 12.1.D says that $\pi_n(\Omega_{\lambda(a)} STF, \lambda) = \pi_{n+1}(STF, \lambda(a))$. One checks, that $\pi_n(STF) = 0, n \geq 2$ for $F \neq S^2, \mathbb{RP}^2$. The exactness of sequence (12.5) implies, that $\pi_n(\Omega STF, \lambda) = 0, n \geq 2$. Using the weak homotopy equivalence, given by the $h$-principle, we get the second statement of the Theorem.

This finishes the proof of Theorem 11.4.A. □
Bibliography

[9] A. V. Inshakov, Invariany tipa J^+,J^- v gladih kryuy na dvumernykh mnogoebrus-