1. [8 points] In lecture Prof Huth told us that the sum of 2 sinusoidal motions can be written as a single sinusoidal motion at the *average* frequency multiplied by an ‘envelope’ function which has maxima occurring at the *difference* frequency:

\[
\sin(\omega_1 t) + \sin(\omega_2 t) = 2 \sin \left( \frac{\omega_1 + \omega_2}{2} t \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t \right)
\]

It is the envelope which causes the *beats* (modulation in amplitude—see graph in p.5 of lecture 4). Even though the envelope looks like beats occur at frequency \((\omega_1 - \omega_2)/2\), two maxima occur per period of the envelope function, so beats actually occur at frequency \(\omega_1 - \omega_2\).

If the minimum beat frequency you can recognize is 1 Hz, the guitarist can tune her A string to within 1 Hz of 110 Hz and her high E string to within 1 Hz of 329.63 Hz. In a worst case scenario, then, her A string might be at 111 Hz (or 109 Hz) while her E string would be at 330.63 Hz (or 328.63 Hz). The respective percentage errors in tuning are \((\frac{1}{110}) \times 100 = 0.9\%\) and \((\frac{1}{329.63}) \times 100 = 0.3\%\).

2. (a) [5 points] ‘Linear superposition’ means that you simply add the displacements caused by the two pulses. Draw the two original pulses travelling along (I used dashed lines for this); their horizontal motion is at the wave speed of 1 unit (square) per second. Then add their vertical displacements where they overlap:

(b) [2 points] The pulse on the right needs to be a leftwards-travelling, *inverted* (i.e. upside-down) version of the pulse on the left. The asymmetric shape needs to be such that the steep side arrives first, just like it does from the left side:
(c) [5 points] We can now add the two pulses from the part b, as they travel along in time, in the same way as part a. Notice that, as predicted, P doesn’t move, so is a valid ‘fixed end’:

![Diagram of wave interference](image)

The shaded region is the part of the string beyond the fixed end, which can now be ignored (it was just used to construct the fixed end reflection). This is very similar to thinking about the ‘image’ formed in a plane mirror (in reality there is nothing behind the mirror!).

[1 point] Upon reflection from the fixed end, the pulse is inverted, and of course now travelling in the opposite direction, with the

(d) [2 points] This could be slightly confusing, since d is more conventionally used to represent longitudinal displacement of air in the tube. However, we asked about if d represented pressure (change from usual atmospheric pressure) in the sound wave. From lecture 6, an open tube acts as a fixed end of pressure, so would correspond to the above graphs.

3. [13 points]

![Diagram of sound wave](image)

[4 points] Figuring out the path difference $\Delta L$ (the extra distance from the lower source to the listener compared to the upper source) was covered in lecture 8. The above shows the little right-angle triangle with angle $\theta$ opposite the side $\Delta L$, and hypotenuse $d = 1.3$ m. So $\Delta L = d \sin \theta$.

[1 point] The wavelength in air is $\lambda = v/f = (340$ m s$^{-1})/(440$ Hz) $\approx 0.773$ m. If you used another close value for $v$, this was fine.

(a) [4 points] For constructive interference (this means when the crests exactly line up, i.e.
the most constructive), \( \Delta L = n\lambda \), where \( n \) is any positive or negative integer. Notice that this only applies because the sources are in phase (other, non-zero, phase differences would change this condition). Since \( |\Delta L| \) cannot exceed \( d \), and \( d \) is less than \( 2\lambda \), the only possible \( n \) values are \( n = -1, 0, +1 \).

- \( n = -1 \): \( d\sin \theta = \Delta L = -\lambda \), solving for \( \theta \) gives \( \theta = \sin^{-1}(-\lambda/d) \approx \sin^{-1}(-0.594) \approx -36.5^\circ \).
- \( n = 0 \): \( \theta = \sin^{-1}(0) = 0^\circ \) (straight ahead direction).
- \( n = +1 \): identical to \( n = -1 \) without the minus signs. \( \theta \approx 36.5^\circ \).

(b) [4 points] For destructive interference (this means when a crest exactly lines up with a trough, i.e., the most destructive), \( \Delta L = (n + \frac{1}{2})\lambda \), where \( n \) is any positive or negative integer. \( d \) is less than \( 5\lambda/2 \), so the possible \( n \) are \( n = -3/2, -1/2, +1/2, +3/2 \).

- \( n = -3/2 \): \( d\sin \theta = \Delta L = -3\lambda/2 \), solving for \( \theta \) gives \( \theta = \sin^{-1}(-3\lambda/2d) \approx \sin^{-1}(-0.892) \approx -63.1^\circ \).
- \( n = -1/2 \): \( \theta = \sin^{-1}(-\lambda/2d) \approx \sin^{-1}(-0.297) \approx -17.3^\circ \).
- \( n = +1/2 \): identical to \( n = -1/2 \) without the minus signs. \( \theta \approx 17.3^\circ \).
- \( n = +3/2 \): identical to \( n = -3/2 \) without the minus signs. \( \theta \approx 63.1^\circ \).

The pattern of constructive (C) and destructive (D) directions is shown on the figure above.

4. [14 points]

(a) [3 points] In these graphs, I labelled displacement by \( d \). These snapshots show a single instant of time (so the function is \( d(x) \)):

(b) [3 points] A crucial fact is needed: in a standing wave, nodes are always separated by \( \lambda/2 \). So we want an exact integer number of half-wavelengths to fit into the space \( L \), so as to have a node at each end. Written mathematically,

\[
L = \frac{n\lambda_n}{2}
\]

You can check this works for \( n = 1, 2 \ldots \) in the graphs above. Therefore,

\[
\lambda_n = \frac{2L}{n}
\]

(c) [2 points] We know that the frequency is given by \( f = \nu/\lambda \), and that \( \nu \) is constant, so we use the \( n^{th} \) wavelength to get the corresponding frequency.

\[
f_n = \frac{\nu}{\lambda_n} = n\frac{\nu}{2L}
\]
(d) [1 point] The pattern of frequencies is a just integer multiples of the fundamental \( (n = 1) \). This is called a harmonic series.

Even and odd refer to the reflection symmetry about a vertical line at \( L/2 \). The mode is even if it’s symmetric about that line, and odd if it is the negative of itself on the other side of the line (this latter case is called anti-symmetric). *’s (for even modes) should be placed by \( n=1, 3, 5 \ldots \) and #’s (for odd modes) should be placed by \( n=2, 4, 6 \ldots \)

Notice that this naming is opposite to that of the harmonics as used in music: ‘odd’ harmonics are \( n = 1, 3, 5 \ldots \), ‘even’ harmonics are \( n = 2, 4 \ldots \)

(e) [1 point] For \( n = 1 \) we know that \( f = 440 \text{ Hz} = v/2L \) so we conclude from part b) that
\[
f_n = (440n) \text{ Hz}
\]

(f) [2 points] Since for the fundamental, \( 440 \text{ Hz} = v/2L \), rearrange to get \( v = (440 \text{ Hz}) \times (2 \times 0.32 \text{ m}) = 281.6 \text{ m/s} \).

(g) [2 points]
- For \( 440 \text{ Hz} \) it’s an A (above middle C).
- For \( 880 \text{ Hz} \) \( (n = 2) \) it’s an A an octave above that (1 octave is always a doubling of frequency).
- \( 1320 \text{ Hz} \) \( (n = 3) \) is E, a perfect fifth above that (this interval corresponds to a 3:2 frequency ratio). Or you could use the chart in the sourcebook.
- \( 1760 \text{ Hz} \) \( (n = 4) \) is A (2 octaves above the original).
- \( 2200 \text{ Hz} \) is C#, a major third above that (this interval corresponds to a 5:4 frequency ratio). Alternatively you could have found that 4 semitones above 1760 Hz is \( (1760 \text{ Hz}) \times (1.06)^4 \approx 2222 \text{ Hz} \), which is not far \( (\approx 1\%) \) from 2200 Hz. Later we will learn why this last calculation does not give the exact 5:4 frequency ratio.