Sci A49: Problem Set 10 SOLUTIONS

Total: 50 points. See accompanying PDF file for table for Qu.2.

1. The strings unstretched length \( L = (1/1.005) \approx 0.995 \) m. It has, thus, stretched about 5 mm, at which point the restoring force is told to us as 1000 N. Assuming this force to be linear in the extra length of the spring, we can say that the average force applied to the string over the five millimeters of stretching is 500 N and so the work done stretching the string is \( 500 \times 0.005 = 2.5 \) Joules. (Note that we could have found this by finding \( k \), spring constant, first, then using \( PE = \frac{1}{2}kx^2 \).) If we assume all of this energy ends up as kinetic energy uniformly spread over the entire string when the string snaps we have

\[
\frac{1}{2}mv^2 = 2.5
\]

with \( m \) the total mass of the string.

If we had the mass of the string we would be done, but we don’t so we need to figure it out. We use

\[
\lambda f = v = \sqrt{\frac{T}{\mu}}
\]

where \( \mu \) is the mass per unit length of the string. We know the string’s tension \( T = 1000 \) N and the wavelength and fundamental frequency of the lowest mode of vibration, \( \lambda = 2 \) meters and \( f = 220 \) Hz, respectively. Solving for \( \mu \), we find \( m = \mu = 5 \) grams.

We can now solve for the speed of the string, obtaining \( v = \sqrt{\frac{5}{0.005}} \approx 31 \) m/s. Since 1 meter ≈ 6.214 × 10⁻⁴ miles and 1 hour is equal to 3600 seconds, this is roughly 69 miles per hour.

2. The first job here is to calculate the fundamental frequencies correctly. In the just tuning, the minor third is at a frequency a factor of 6/5 times the fundamental, so C (just) has a frequency of 264 Hz. The equal-tempered C has a frequency of \( 220(2^{3/12}) = 262 \) Hz. The Pythagorean D has a frequency of \( 220(4/3) = 293 \) Hz, while the equal-tempered D has a frequency of \( 220(2^{6/12}) = 294 \) Hz. The equal-tempered tritone and major-seventh have frequencies of \( 220(2^{5/12}) = 311 \) Hz and \( 220(2^{11/12}) = 415 \) Hz, respectively.

Once we have these frequencies we simply build up the harmonics by successive multiplication by the integers 1, 2, 3 …. The results are in the table (PDF file available in this directory). Then we just compare the frequencies in one column to those in the column for our A(220). Note that we are concerned with any frequency pairs which fall within a certain range, not simply those which happen to correspond to the same harmonic number. (This number, after all, is a purely notational construct.) The table shows that the tritone sounds worst, followed by the major seventh and then the fourth and then the minor third. Note that there is some room for interpretation here, for instance, is a single 25 Hz dissonance worse than three 75 Hz dissonances? We let you decide. We also see that the equal-tempered third is considerably worse than the just third, although the equal tempered fourth is no worse than the Pythagorean fourth because the fundamental frequency change was so small.

3. You have to look for patterns in the frequencies, in particular close to integer ratios. Notice 508 : 760 : 1011 : 1268 is very nearly 2 : 3 : 4 : 5 with a fundamental of about 253 Hz. This is the perceived frequency, somewhere between a B₃ and a C₄. It is, of course, not actually present. This is just the illusion of the ‘missing fundamental’ demonstrated in lecture 19. Note that the lowest component 161 Hz is not the fundamental.
4. (a) At the simplest level (ignoring, among other things, the slight anharmonicity due to the thickness of the piano wire that we talked about in class and the use of more than one slightly detuned strings for certain notes) the piano is no different from a guitar or violin: although the soundboard resonates in complicated ways, it is analogous to the body of the violin and it is the strings in each case which determine the frequencies that you see in the output spectrum. Take a look, also, at Rossing Figure 14.5 where the very nearly harmonic quality of the output sound is striking. This sort of reasoning would lead one to suspect that the signal produced is periodic. On the other hand, the piano’s great complexity means that the simplest level is not such a good approximation here. The piano string is struck once and the sound quality immediately begins to change, continuing to develop throughout the duration of the note. Says Rossing, (p. 293) “the sound spectrum of the piano is constantly changing with time, as in the case of…percussion instruments...” This rapid variation of the sound prevents us from accurately calling the signal truly periodic. (In fact, the variation of the signal over timescales small compared to the total duration prevents us from calling the signal roughly periodic.)

(b) Bowing a gong, like the chladni plates in lab, produces continual excitation of a single resonance, producing a clear pitch and a periodic signal.

(c) That we make use of how the spectrum changes over time, paying particular attention to the order in which the changes occur. More specifically, we pay particular attention to the attack transients of the sound.

5. (a) We use \( Q = \omega \tau /2 \) with \( \omega = 2\pi f \) to find \( Q = 1885 \).
(b) Recall that \( Q = f / \Delta f \). With \( f = 300 \text{ Hz} \) and \( Q = 1885 \), we solve for \( \Delta f = 0.16 \text{ Hz} \). So we can be off by only about 0.08 Hz in either direction.
Note that this frequency width (0.16 Hz) is just the inverse of the decay time divided by \( 2\pi \).
Note also that, although this difference is about 1 cent (0.06% change), which happens to be about the tuning difference between the 3 strings on many of the piano’s notes, we did not expect you to discuss the effect of these slightly-detuned 3 strings (this would be very complicated).