1. Given the matrix \( A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \),

(a) \( \det(A - \lambda I) = (4 - \lambda)(1 - \lambda) + 2 = (\lambda - 3)(\lambda - 2) \). Eigenvalues are \( \lambda = 2, 3 \).

(b) \( \lambda = 2 \): find null-space vectors in \( (A - 2I) \).
\( \lambda = 3 \): find null-space vectors in \( (A - 3I) \).

Each time, you should find one free variable. (If you don’t find any, you know \( \lambda \) is not a correct eigenvalue).

\[
\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\]

(c) \( P = \) eigenvectors stacked as columns. \( D = \) eigenvalues. Note you must match up the order of eigenvectors with eigenvalues. You must also find the inverse of \( P \) - be careful with signs since \( \det P \) is either +1 or -1 depending on your choice of eigenvectors. For example,

\[
A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.
\]

2. (a) It can be diagonalizable, but only if the dimension of the eigenspace corresponding to the (multiplicity 2) eigenvalue \( \lambda = -2 \) is also 2. Then 3 lin. indep. eigenvectors exist.

(b) The original vector’s length is \( \sqrt{1^2 + 3} = 2 \). So divide both entries by this length: \( \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \). This vector now is a unit vector.

(c) Always true, since each distinct eigenvalue must have an eigenvector, and we showed this set is lin. indep., so can be used for diagonalization. I think some of you thought that for \( B \) an \( n \times n \)
matrix, if there were less than \( n \) real eigenvalues, then \( B \) is not diagonalizable. This is not true. Counting complex eigenvalues, there are always \( n \) eigenvalues (roots of characteristic equation). If they are distinct, then the above follows.

(d) Eigenvectors found via \((A - \lambda I)v = 0\):

\[
\begin{bmatrix}
-1 - 3i & 5 \\
-2 & 1 - 3i
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

Since there will be one free variable (because a \( 2 \times 2 \) matrix has two eigenvalues, and the other one must therefore be the conjugate, \( 2 - 3i \)), you can use either row to give the vector (as in book). This trick only works for the \( 2 \times 2 \) case.

\[e.g. \text{second row: } -2v_1 + (1 - 3i)v_2 = 0 \text{ gives } v = \begin{bmatrix} 1 - 3i \2 \end{bmatrix}.\]

Careful with signs!

\[v = \begin{bmatrix} 5 \\
1 + 3i
\end{bmatrix}\]
is also correct (they are parallel, even though they don’t look it! The complex arithmetic to show this is not hard).

(e) It is a repellor because both eigenvalues are larger than 1. (Strictly, have magnitudes larger than 1). How to remember? Taking higher powers of numbers which are larger than 1 results in heading off to infinity, \( i.e. \) repelling away from the origin.

3. (a) Inner product \( u \cdot y = 10 \), which is not 0, so they are not orthogonal.

(b) orthogonal projection

\[
\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{10}{5} \begin{bmatrix} 1 \\
2
\end{bmatrix} = \begin{bmatrix} 2 \\
4
\end{bmatrix}.
\]

This is the vector you get by dropping a perpendicular from the location \( y \) to the direction given by \( u \). Note that the answer is a scalar multiple of \( u \) (not \( y \)), even though it is called \( mbfy \). It’s easy to remember this when you draw the projection diagram.