1. [9 points] Solve the linear system $Ax = b$ given

$$A = \begin{bmatrix} 2 & -6 & 1 & -2 \\ -1 & 3 & 2 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(a) If inconsistent, explain why. If consistent, write the general solution in parametric vector form (i.e. in the form $x = p + su \cdots$ etc):

$$x =$$

(b) Write in the same form the solution to the corresponding homogeneous problem $Ax = 0$:

$$x =$$
2. [10 points]

(a) Define the concept of linear independence.

(b) Fixing $h = 0$, is the set of three vectors
\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
3 \\
2 \\
h
\end{bmatrix},
\]
linearly independent?

(c) Still fixing $h = 0$, is the last of these vectors in the span of the first two vectors?

(d) What condition on $h$ is required if the set of three vectors is to span $\mathbb{R}^3$?
3. [6 points] A linear transformation $T$ can be written as a function $T(x_1, x_2) = (3x_2, x_1 - x_2, 2x_1)$.

(a) Find the *standard matrix* for this transformation (check you have the correct size matrix):

$$A =$$

(b) Is $T$ onto $\mathbb{R}^3$? (why?)

4. [8 points]

(a) A matrix $A$ has been factored into $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$  

Use this factorization to solve $Ax = b$, for the case $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$:
(b) Is the matrix $A$ invertible? (why?)

5. [7 points]

(a) True/false: if a system of linear equations has two different solutions, then it must have infinitely many solutions.

(b) True/false: a set of three vectors in $\mathbb{R}^2$ can be linearly independent. (Explain your answer)

(c) A transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps $(1, 0)$ to $(3, 4)$ and $(2, 0)$ to $(6, 7)$. What (if anything) can we say about whether $T$ is a linear transformation?
6. [9 points] Consider a (dubious) economy with two sectors, pizza (P) and beer (B). Let \( x_1 \) be the units of output (i.e. dollars of production) by the pizza sector, and \( x_2 \) be the units of output by the beer sector. Recall that Leontief’s equation is \( \mathbf{x} = \mathbf{C}\mathbf{x} + \mathbf{d} \). Suppose the consumption matrix is \( \mathbf{C} = \begin{bmatrix} 2/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix} \).

(a) How much intermediate demand for beer is created by each unit production of pizza? (Hint: you do not need to solve anything!)

(b) What output vector \( \mathbf{x} \) satisfies a final demand \( \mathbf{d} \) of zero units of pizza and 4 units of beer?

7. [11 points]

(a) Find the determinant of the following matrix, taking care to get the correct sign. You may use any method [hint: one is much
easier than the other]. State which you use.

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 4 \\
2 & 3 & 3 \\
\end{bmatrix}
\]

(b) Use your above result to find the determinant of this matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 7 & 1 & 1 \\
2 & -9 & 2 & 4 \\
2 & 5 & 3 & 3 \\
\end{bmatrix}
\]

(c) Prove that for \( n \times n \) matrices \( A \) and \( B \), where \( A \) is invertible,
\[
\det(A^{-1}BA) = \det B.
\]

(d) True/false: If a \( n \times n \) matrix has determinant zero, its rows can span \( \mathbb{R}^n \)?