1. [17 points] A set of vectors in $\mathbb{R}^3$ is given by \( W = \left\{ \begin{bmatrix} a + 2b + 2c \\ -2b + c \\ a + 3c \end{bmatrix} \right\} a, b, c \text{ real} \). 

   (a) [4 points] Determine, using the tests for a subspace, whether $W$ is a subspace of $\mathbb{R}^3$. (Explain any claims you make).

   (b) [4 points] Find a basis for $W$. 

   (c) [4 points] Show that $W$ is a subspace of $\mathbb{R}^3$. (Explain any claims you make).
(c) [2 points] What is dim $W$?

(d) [3 points] Give a definition of the set $W^\perp$.

(e) [4 points] Find a basis for $W^\perp$.

2. [14 points] Consider the symmetric matrix $A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$, which has eigenvalues $\lambda = -3, 7$.

(a) [4 points] Find the eigenvectors.
(b) [4 points] Give a formula for $x_k$, the $k^{th}$ iterate of the discrete dynamical system $x_k = Ax_{k-1}$, with initial vector $x_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Your formula should involve only numbers and $k$.

(c) [1 point] To which direction will $x_k$ tend in the limit $k \to \infty$?

(d) [2 points] Classify the quadratic form $Q(x) = 2x_1^2 + 10x_1x_2 + 2x_2^2$ (is it positive/negative definite, or indefinite?)

(e) [3 points] If $P$ is the matrix of normalized eigenvectors of $A$ stacked in columns, and we change variable to $y = P^T x$, express the above quadratic form $Q$ as a function of $y_1$ and $y_2$, the components of $y$. 
3. [12 points]

(a) [2 points] Do the vectors \( \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \) and \( \mathbf{x}_2 = \begin{bmatrix} 0 \\ 15 \\ -2 \end{bmatrix} \) form an orthogonal set?

(b) [6 points] Construct an orthonormal basis \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) for \( \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \), with \( \mathbf{x}_1, \mathbf{x}_2 \) as given above. [Hint: one way round is much easier than the other!]

(c) [4 points] Now consider a general square \( n \times n \) matrix \( A \) whose columns form an orthonormal set. Prove the remarkable result that the rows of \( A \) also form an orthonormal set. [Hint: consider the product \( A^T A \).]
4. [16 points] In parts a–c you don’t necessarily need to diagonalize the matrix in order to answer the question.

(a) [3 points] Is the matrix \( A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \) diagonalizable? Why?

(b) [4 points] Is the matrix \( B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \) diagonalizable?

(c) [4 points] Is the matrix \( C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \) diagonalizable?
(d) [5 points] Find a matrix \( P \) and a diagonal matrix \( D \) such that the above matrix can be written \( C = PDP^{-1} \).

\[
P = \begin{bmatrix} \quad \end{bmatrix} \quad \quad \quad D = \begin{bmatrix} \quad \end{bmatrix}
\]

5. [10 points] You are given \( A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} \).

(a) [6 points] Find \( \hat{x} \), the least squares solution of the inconsistent system \( Ax = b \). [Hint: make sure you notice a common factor which simplifies the arithmetic, otherwise re-check your work!]
(b) [4 points] What is the least squares error associated with your solution?

6. [10 points]

(a) [5 points] \( \mathcal{B} = \{1, 1+2t, -1+t^2\} \) is a basis for \( \mathbb{P}_2 \), the vector space of all degree-2 polynomials. Find the \( \mathcal{B} \)-coordinate vector \([p]_\mathcal{B}\) of the polynomial \( p(t) = 1 + 2t + 3t^2 \).

(b) [5 points] \( \mathcal{C} = \{c_1, c_2\} \) is a basis for \( \mathbb{R}^2 \), with \( c_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and
\( c_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Also \( \mathcal{B} = \{b_1, b_2\} \) is another basis for \( \mathbb{R}^2 \) defined by \( b_1 = 2c_1 - 2c_2 \) and \( b_2 = c_1 + 2c_2 \). Find the change of coordinates matrix \( P_{\mathcal{C} \rightarrow \mathcal{B}} \) such that for any vector \( x \) we have \([x]_\mathcal{C} = P_{\mathcal{C} \rightarrow \mathcal{B}} [x]_\mathcal{B} \).

\[
P_{\mathcal{C} \rightarrow \mathcal{B}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \end{bmatrix}
\]
7. [11 points]

(a) [1 point] True/false: two eigenvectors belonging to different eigenvalues are always linearly independent?

(b) [1 point] True/false: two eigenvectors belonging to the same eigenvalue are always linearly dependent?

(c) [1 point] True/false: The steady-state (long-time limit) vector of a Markov chain with stochastic matrix \( A \) is an eigenvector of \( A \) with eigenvalue 1?

(d) [1 point] True/false: A symmetric matrix is always diagonalizable even if its eigenvalues are not distinct?

(e) [2 points] A real 3 \( \times \) 3 matrix has eigenvalues 2 and \(-4 - i\). Are there any more eigenvalues, if so of what value?

(f) [2 points] What is the rank of a 5 \( \times \) 3 matrix if a basis for its null space contains only 1 vector?

(g) [1 point] What is the largest possible dimension of the row space of a matrix of size 7 \( \times \) 4?

(h) [2 points] True/false: The product of 2 symmetric matrices is always itself symmetric? [Hint: try examples if stuck].