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- Define the \( m \)\(^{th} \) partial sum of an infinite series as:

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s_m = \sum_{n=1}^{m} a_n
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- Define the \(m^{th}\) partial sum of an infinite series as:

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- A series converges if the sequence of partial sums \(\{s_m\}\) converges. Otherwise the series diverges.
One easy consequence of this definition is the test for divergence:

Consider a series \( \sum_{i=k}^{\infty} a_n \) and the sequence of summands, \( \{a_n\} \). If \( a_n \) converges to \( L \neq 0 \) then \( \sum_{i=k}^{\infty} a_n \) diverges.
The test for divergence

It is **NOT TRUE** that

\[
\text{If } a_n \to 0 \text{ then } \sum_{i=k}^{\infty} a_n \text{ converges.}
\]

The harmonic series is a counterexample:

\[
\frac{1}{n} \to 0
\]

but

\[
\sum_{i=1}^{\infty} \frac{1}{n} \text{ diverges}
\]
Examples

- We’ve seen two examples via improper integrals:

\[ \sum_{i=1}^{\infty} \frac{1}{n} \] diverges

and

\[ \sum_{i=1}^{\infty} \frac{1}{n^2} \] converges
More examples

• Does
  \[ \sum_{i=2}^{\infty} \frac{1}{n^2 - n} \]
  converge or diverge?

• Let \( a \) be a real number. For which values of \( r \) does
  \[ \sum_{i=0}^{\infty} ar^n \]
  converge?
Suppose \( a \) is a real number and \( |r| < 1 \) then

\[
\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}
\]
Geometric series examples

\[
\sum_{i=1}^{\infty} 10 \frac{3^n}{4n-1}
\]

\[
\sum_{i=0}^{\infty} -2 \frac{1}{4^n}
\]

\[
\sum_{i=2}^{\infty} \frac{9^n-1}{10^{n-1}}
\]
We’ve already used integrals to help determine the convergence of some series. We can now formalize this into a test:

Suppose $f$ is a continuous, positive, decreasing function defined for $1 \leq x < \infty$ and let $a_n = f(n)$. Then

- If $\int_1^\infty f(x) \, dx$ converges, then $\sum_{i=1}^\infty a_n$ converges as well.
- If $\int_1^\infty f(x) \, dx$ diverges, then $\sum_{i=1}^\infty a_n$ diverges as well.
Examples

• Suppose $p > 0$, for which values of $p$ does

$$\sum_{i=1}^{\infty} \frac{1}{n^p}$$

converge?

•

$$\sum_{i=1}^{\infty} \frac{\ln(n)}{n}$$

•

$$\sum_{i=1}^{\infty} ne^{-n}$$