We now have quite a bit of experience with integrals of the form:

$$\int_a^b f(x) \, dx$$

But, what happens if we let either one or both of $a$ and $b$ become infinite?
We now have quite a bit of experience with integrals of the form:

\[ \int_{a}^{b} f(x) \, dx \]

But, what happens if we let either one or both of \( a \) and \( b \) become infinite?

These are called *improper integrals* and require careful handling.

Standard example: \( f(x) = \frac{1}{x} \)

We know:

\[ \int_{a}^{b} \frac{1}{x} \, dx = \ln(b) - \ln(a) \]
• Letting $a = 1, b = t$ we have

$$A(t) = \int_{1}^{t} \frac{1}{x} \, dx = \ln(t) - \ln(1) = \ln(t)$$

• \[
\lim_{t \to \infty} A(t) = \infty
\]

• In other words, as $t$ grows, the area under this curve tends to $\infty$. 
So, perhaps all such integrals tend to $\pm \infty$, i.e. they are *divergent* integrals.
So, perhaps all such integrals tend to $\pm \infty$, i.e. they are divergent integrals.

- Consider

$$A(t) = \int_{1}^{t} \frac{1}{x^2} \, dx$$

$$= -\frac{1}{x} \bigg|_{1}^{t}$$

$$= -\frac{1}{t} + 1$$

- As $t \to \infty$, $A(t) \to 1$. 
Improper Integrals

So, we define the value of an integral of the form

$$\int_{a}^{\infty} f(x) \, dx$$

to be

$$\lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

If the limit tends to $\pm \infty$, we say the integral \textit{diverges}. If, instead, it tends to a finite value, we say the integral \textit{converges} to that value. Similarly,

we define the value of an integral of the form

$$\int_{-\infty}^{b} f(x) \, dx$$

to be

$$\lim_{t \to -\infty} \int_{t}^{b} f(x) \, dx$$
Improper Integrals

And, if both bounds are infinite:

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx
\]

Examples:

- \[
\int_{1}^{\infty} \frac{1}{(3x + 1)^2} \, dx
\]
- \[
\int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, dx
\]
- \[
\int_{0}^{\infty} \cos^2(\theta) \, d\theta
\]
Another type of improper integral: integrals where the integrands have discontinuities. Example:

\[ \int_{0}^{3} \frac{dx}{x - 1} = \lim_{t \to 1} \left( \int_{0}^{t} \frac{dx}{x - 1} + \int_{t}^{3} \frac{dx}{x - 1} \right) \]