Math 8, Winter 2005

Scott Pauls
Dartmouth College, Department of Mathematics
2/25/05
A function of two variables, $f$, is a rule that assigns to each vector $<x, y> \in D \subset \mathbb{R}^2$ a real number denoted by $f(x, y)$. The set $D$ is called the domain of $f$ and its range is the set of values that $f$ takes on, i.e. $\{t\}$ where $f(x, y) = t$ for some $<x, y> \in D$.

Examples:

- $f(x, y) = x^2 + y^2$
- $f(x, y) = \sin(xy)$
- $f(x, y) = \sqrt{1 - x^2 - y^2}$
Plotting

For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  
  $$x^2 + y^2 = k$$

  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  
  $$x^2 + y^2 = k$$

  has no solutions.

- For $k = 0$
  
  $$x^2 + y^2 = 0$$

  consists of the single point $(0, 0)$. 
Plotting

**contour plots**
For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[ x^2 + y^2 = k \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[ x^2 + y^2 = k \]
  has no solutions.

- For \( k = 0 \)
  \[ x^2 + y^2 = 0 \]
  consists of the single point \((0, 0)\).
contour plots

For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  \[ x^2 + y^2 = k \]
  
  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  \[ x^2 + y^2 = k \]
  
  has no solutions.

- For $k = 0$
  \[ x^2 + y^2 = 0 \]

  consists of the single point $(0, 0)$. 
Plotting

**contour plots**
For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  
  \[ x^2 + y^2 = k \]
  
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  
  \[ x^2 + y^2 = k \]
  
  has no solutions.

- For \( k = 0 \)
  
  \[ x^2 + y^2 = 0 \]
  
  consists of the single point \( (0, 0) \).
contour plots

For $f(x, y)$, graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$

  $$ x^2 + y^2 = k $$

  is a circle of radius $\sqrt{k}$.

- For $k < 0$

  $$ x^2 + y^2 = k $$

  has no solutions.

- For $k = 0$

  $$ x^2 + y^2 = 0 $$

  consists of the single point $(0, 0)$. 
Plotting

contour plots
For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.
Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  \[ x^2 + y^2 = k \]
  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  \[ x^2 + y^2 = k \]
  has no solutions.

- For $k = 0$
  \[ x^2 + y^2 = 0 \]
  consists of the single point $(0, 0)$. 
For $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  \[ x^2 + y^2 = k \]
  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  \[ x^2 + y^2 = k \]
  has no solutions.

- For $k = 0$
  \[ x^2 + y^2 = 0 \]
  consists of the single point $(0, 0)$. 
Plotting

**contour plots**

For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

**Example:** $f(x, y) = x^2 + y^2$

- For $k > 0$
  
  \[ x^2 + y^2 = k \]

  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  
  \[ x^2 + y^2 = k \]

  has no solutions.

- For $k = 0$
  
  \[ x^2 + y^2 = 0 \]

  consists of the single point $(0, 0)$. 
Plotting

Contour plots

For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
contour plots

For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
contour plots

For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  
  $$x^2 + y^2 = k$$

  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  
  $$x^2 + y^2 = k$$

  has no solutions.

- For $k = 0$
  
  $$x^2 + y^2 = 0$$

  consists of the single point $(0, 0)$. 
Plotting

contour plots

For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
contour plots
For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
Plotting

contour plots

For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.

Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
Plotting

contour plots

For $f(x, y)$ graph $f(x, y) = k$ for different values of $k$ and put together in a graph.

Example: $f(x, y) = x^2 + y^2$

- For $k > 0$
  
  $$x^2 + y^2 = k$$

  is a circle of radius $\sqrt{k}$.

- For $k < 0$
  
  $$x^2 + y^2 = k$$

  has no solutions.

- For $k = 0$
  
  $$x^2 + y^2 = 0$$

  consists of the single point $(0, 0)$. 
Plotting

contour plots
For \( f(x, y) \) graph \( f(x, y) = k \) for different values of \( k \) and put together in a graph.
Example: \( f(x, y) = x^2 + y^2 \)

- For \( k > 0 \)
  \[
x^2 + y^2 = k
  \]
  is a circle of radius \( \sqrt{k} \).

- For \( k < 0 \)
  \[
x^2 + y^2 = k
  \]
  has no solutions.

- For \( k = 0 \)
  \[
x^2 + y^2 = 0
  \]
  consists of the single point \((0, 0)\).
Limits in more than one variable are much harder than in a single variable.

Let $f$ be a function of two variables. Then,

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

if, given an $\varepsilon > 0$ there is a $\delta > 0$ so that if the distance between $(x,y)$ and $(x_0,y_0)$ is less than $\delta$ then

$$|f(x,y) - L| < \varepsilon$$
Proving a limit exists is difficult, but sometimes showing one does not exists is easier.

- Look at the function restricted to different lines through \((x_0, y_0)\).
- If the limit along one line is different from the limit along a different line, then the limit does not exist.
- Example:

\[
f(x, y) = \frac{xy^2}{x^2 + y^2}
\]
A function of two variables $f$ is *continuous* at $(x_0, y_0)$ if

$$\lim_{(x,y) \to (x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Examples:

- Polynomials
- Rational functions: discontinuities when the denominator is zero