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Volume

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- We understand how to compute area, so let’s slice the solid into 2-dimensional sheets. Calculate the area for that slice, \( A(x) \).

- Thicken the slice to a slab of width \( \Delta x \), then the volume of the slab is \( \text{approximately} \ A(x)\Delta x \).
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Difficulty: Compute \(A(x)\).
Examples

- Sphere: $x^2 + y^2 + z^2 = 1$
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- Same problem but rotate about the y-axis.
• Find the volume of a right circular cone with height $h$ and radius $r$. 
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• Find the volume of a solid whose base $S$ is the parabolic region $\{(x, y) | x^2 \leq y \leq 1\}$ and whose cross-sections perpendicular to the y-axis are equilateral triangles.
Integration techniques
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• Where does substitution come from: the chain rule!

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• Integrate both sides to get:

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f(g(x)) = \int_a^b \frac{d}{dx} f(g(x)) \, dx = \int_a^b f'(g(x))g'(x) \, dx
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Integration techniques

- Every differentiation rule gives an integration rule.
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Integration by parts

- Rearrange terms to get the *integration by parts* formula:

\[ \int_{a}^{b} f(x)g'(x) \, dx = f(x)g(x)|_{a}^{b} - \int_{a}^{b} f'(x)g(x) \, dx \]

or, letting \( u = f(x) \), \( v = g(x) \),

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• Key: pick correct \( u \) and \( dv \)
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- Generalization: Differentiation makes things simpler, integration makes things more complicated.

- Key is simplicity of resulting integral: pick $u$ so that $du$ is simpler, pick $dv$ so that $v$ is at least not much worse than $dv$. 
Integration by parts

• \( \int xe^x \, dx \)

• \( \int x^2 \sin(x) \, dx \)

• \( \int \arctan(x) \, dx \)

• \( \int \ln(x) \, dx \)

• \( \int e^x \sin(x) \, dx \)
Rules of thumb

- Choices for $u$: polynomials, arc-trig functions, logarithms, $\sin(x)$, $\cos(x)$
- Choices for $dv$: $e^x$, $\sin(x)$, $\cos(x)$, polynomials