Multiplication of vectors

If \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) and \( \vec{w} = \langle w_1, w_2, w_3 \rangle \) are two vectors, then the dot product of \( \vec{v} \) and \( \vec{w} \) is

\[
\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3
\]

Note: The dot product is sometimes also called the scalar product or inner product.
Properties of the dot product

1. The dot product calculates lengths:

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

2. The dot product calculates the angle between vectors:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$$

where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$. We can also rewrite this formula as:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$$

3. We say two vectors are orthogonal (or perpendicular) if $\theta = \frac{\pi}{2}$ and that the two vectors are parallel if $\theta = 0$.

4. From this, we see that two vectors are orthogonal if

$$\vec{v} \cdot \vec{w} = 0$$
Find the cosine of the angle between the following vectors:

- $<3, 4>$ and $<5, 12>$
- $<1, 2, 3>$ and $<4, 0, -1>$
- Find a vector that is perpendicular to both $<-1, 1, 0>$ and $<4, 0, -1>$. 

Proof of the angle formula

This follows from the law of cosines: If a triangle is determined by points $O$, $A$ and $B$ and $\theta$ is the angle at the vertex $O$ then

$$|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}| \cos(\theta)$$

If $\vec{v} = \overrightarrow{OA}$ and $\vec{w} = \overrightarrow{OB}$ then $\vec{v} - \vec{w}$ is a copy of $\overrightarrow{AB}$. So, the formula becomes

$$|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos(\theta)$$

$$-|\vec{v} - \vec{w}|^2 + |\vec{v}|^2 + |\vec{w}|^2 = 2|\vec{v}||\vec{w}| \cos(\theta)$$

$$|\vec{v}|^2 + |\vec{w}|^2 - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = 2|\vec{v}||\vec{w}| \cos(\theta)$$

$$|\vec{v}|^2 + |\vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2 + 2\vec{v} \cdot \vec{w} = 2|\vec{v}||\vec{w}| \cos(\theta)$$

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$$
It is often useful to be able to project one vector onto another. We have two formulas that help us calculate such a projection:

\[ \text{proj}_{\vec{w}} \vec{v} = \left( \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \right) \vec{v} \]

The scalar projection, or component, of \( \vec{w} \) onto \( \vec{v} \) is

\[ \text{comp}_{\vec{v}} \vec{w} = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|} \]
Examples

- Find the scalar and vector projections of \( \vec{w} = \langle 1, 1, 2 \rangle \) onto \( \vec{v} = \langle -2, 3, 1 \rangle \).

- Let \( T \) be a triangle with vertices at \( P = (1, 0, 0), Q = (0, 1, 0) \) and \( R = (0, 1, 1) \). What is the cosine of the angle between the side \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \)?
There is another way to multiply vectors: given two vectors \( \vec{v} = \langle v_1, v_2, v_3 \rangle \) and \( \vec{w} = \langle w_1, w_2, w_3 \rangle \), their \textit{cross product} is

\[
\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle
\]

Note: we can also phrase this formula in terms of determinants:

\[
\vec{v} \times \vec{w} = \text{det} \begin{pmatrix}
\vec{i} & \vec{j} & \vec{k} \\
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3
\end{pmatrix}
\]
• The vector $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

• Which way? Use the right hand rule.

$$|\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\theta)$$

• Two nonzero vectors are parallel if $|\vec{v} \times \vec{w}| = 0$

• $|\vec{v} \times \vec{w}|$ is equal to the area of a the parallelogram determined by $\vec{v}$ and $\vec{w}$.

• The volume of the parallelopiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$|\vec{u} \cdot (\vec{v} \times \vec{w})|$$