There are several ways to make new power series out of old ones:

Integration:
\[
\int \sum_{n=0}^\infty c_n(x - a)^n \, dx = C + \sum_{n=0}^\infty \frac{c_n}{n+1} (x - a)^{n+1}
\]

Differentiation:
\[
\frac{d}{dx} \sum_{n=0}^\infty c_n(x - a)^n \, dx = \sum_{n=1}^\infty nc_n(x - a)^{n-1}
\]

Substitution: for example, let \( x = \alpha u + \beta \)
\[
\sum_{n=0}^\infty c_n(x - a)^n \, dx = \sum_{n=0}^\infty c_n(\alpha u + \beta - a)^n = \sum_{n=0}^\infty c_n\alpha^n \left(u - \frac{\beta - a}{\alpha}\right)^n
\]
Integration and Differentiation do not change the radius of convergence, but substitution may!

Suppose \( \sum_{n=0}^{\infty} c_n (x-a)^n \) has radius of convergence \( R \), i.e.

\[
\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = R
\]

Then, after substituting \( x = \alpha u + \beta \), we have the series:

\[
\sum_{n=0}^{\infty} c_n \alpha^n \left( u - \frac{\beta - a}{\alpha} \right)^n
\]

Performing the ratio test yields:

\[
\lim_{n \to \infty} \left| \frac{c_{n+1} \alpha^{n+1} \left( u - \frac{\beta - a}{\alpha} \right)^{n+1}}{c_n \alpha^n \left( u - \frac{\beta - a}{\alpha} \right)^n} \right| < 1
\]

\[
\lim_{n \to \infty} \left| \frac{c_{n+1} \alpha}{c_n} \right| \left| \left( u - \frac{\beta - a}{\alpha} \right) \right| < 1
\]
Substitution (con’t)

\[
\lim_{{n \to \infty}} \left| \frac{c_{n+1} \alpha}{c_n} \right| \left| \left( u - \frac{\beta - a}{\alpha} \right) \right| < 1
\]

\[
\left| \left( u - \frac{\beta - a}{\alpha} \right) \right| < \frac{1}{\alpha} \lim_{{n \to \infty}} \left| \frac{c_n}{c_{n+1}} \right|
\]

\[
= \frac{R}{\alpha}
\]

So the radius of convergence of the new series after substitution is \( \frac{R}{\alpha} \).
Power series define functions on their intervals of convergence. Can we find a different description for the function?

\[ \sum_{n=0}^{\infty} c_n (x - a)^n = f(x) \]

Or, given a function, can we find a power series representation?

\[ f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \]
Example: geometric series

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n
\]

Use this to find a power series representation for

- \( f(x) = \frac{1}{(1 - x)^2} \)
- \( f(x) = \ln(1 - x) \)
- \( f(x) = \frac{1}{1 + 2x^2} \)
What about a function like

\[ f(x) = \sin(x) \]

Idea:

- Goal: a power series representation about \( a \)
- The \( m^{th} \) partial sum of a series

\[
\sum_{n=0}^{\infty} c_n (x - a)^n
\]

is a polynomial of degree \( m \):

\[
s_m = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_m(x - a)^m
\]

- Find polynomials that closely match the function, \( f(x) \), near \( x = a \).
\[ m = 0 \text{ and } m = 1 \]

**m = 0**

\[ s_0 = c_0 \] so simply pick \( c_0 = f(a) \).

**m = 1**

\[ s_1 = c_0 + c_1(x - a) = f(a) + c_1(x - a) \]

Find \( c_1 \) so that \( f(x) - (f(a) + c_1(x - a)) \) is as small as possible near \( x = a \):

\[
\frac{(f(x) - f(a))}{x - a} + c_1 \sim f'(a) + c_1
\]

when \( x \) is close to \( a \). So pick \( c_1 = f'(a) \).
Using these values, what is \( s_1 = f(a) + f'(a)(x - a) \)?
Using these values, what is $s_1 = f(a) + f'(a)(x - a)$?

It is just the tangent line to $f$ at $x = a$!