LECTURE OUTLINE
Partial Fractions and Improper Integral

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Math 8
Sep 29, 2004
Goals

SERIOUS Algebra "Review"
Partial Fraction Decomposition
Improper Integrals
Polynomials

$P(x)$ is polynomial with real coefficients provided

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

where the $a_k$ are real numbers. $P(x)$ is said to have degree $n$ provided $a_n \neq 0$. 
Division

Division Theorem: If $P(x)$ and $Q(x)$ are nonzero polynomials with real coefficients, then there exist polynomials $S(x)$ and $R(x)$ such that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

with the degree of $R(x)$ less than the degree of $Q(x)$. 
Example 1

Let

\[ P(x) = x^6 - 3x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 4 \]

and

\[ Q(x) = x^4 - 4x^3 + 8x^2 - 8x + 3. \]

Divide \( P(x) \) by \( Q(x) \).
Factoring a Polynomial

Factorization Theorem: Every polynomial $Q(x)$ with real coefficients, is equal to

$$c(x - r_1)^{N_1} \ldots (x - r_n)^{N_n} (x - a_1)^2 + b_1^2)^{M_1} \ldots ((x - a_m)^2 + b_m^2)^{M_m}$$

for some real numbers $c$, $r_i$, $a_i$ and $b_i$ such that $(x - r_i) \neq (x - r_j)$ when $i \neq j$ and $((x - a_i)^2 + b_i^2) \neq ((x - a_j)^2 + b_j^2)$ when $i \neq j$. 
Example 2

Factor the polynomial

\[ Q(x) = x^4 - 4x^3 + 8x^2 - 8x + 3. \]
Holy Moly! what a mess!

Partial Fraction Decomposition: A rational function $R(x)/Q(x)$ with $Q(x)$ equal to

$$(x - r_1)^{N_1} \ldots (x - r_n)^{N_n} ((x - a_1)^2 + b_1^2)^{M_1} \ldots ((x - a_m)^2 + b_m^2)^{M_m}$$

can be expressed as

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{n} \left( \sum_{k=1}^{N_i} \frac{A_{k,i}}{(x - r_i)^k} \right) + \sum_{i=1}^{m} \left( \sum_{k=1}^{M_i} \frac{B_{k,i}x + C_{k,i}}{((x - a_i)^2 + b_i^2)^k} \right)$$

for some real numbers $A_{k,i}, B_{k,i}$ and $C_{k,i}$. 
Example 3

Find the partial fraction decomposition of

\[
\frac{x + 1}{x^4 - 4x^3 + 8x^2 - 8x + 3}.
\]
Using this Decomposition

This reduces integrating $\frac{P(x)}{Q(x)}$ to integrating functions in the form $x^n$,

$$1 \quad \frac{1}{(x - r)^n},$$

or

$$1 \quad \frac{1}{((x - a)^2 + b^2)^m}.$$

You can do these integrals!
Example 4

Find an anti-derivative of

\[
x^6 - 3x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 4
\]

\[
\frac{x^6 - 3x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 4}{x^4 - 4x^3 + 8x^2 - 8x + 3}
\].
Improper Integral

If $\int_{a}^{t} f(x) \, dx$ exists for every number $t \geq a$, then

$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

provided this limit exists (as a finite number). We say the integral is *convergent* if the limit exist and *divergent* otherwise.
Example 5

Find

$$
\int_{1}^{\infty} \frac{1}{x^p} \, dx.
$$
If \( f(x) \) is continuous on \([a, b)\) and is discontinuous at \(b\), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to b} \int_{a}^{t} f(x) \, dx
\]

provided this limit exists (as a finite number).
Example 6

Find

\[ \int_{0}^{1} \frac{1}{(1 - x)^p} \, dx. \]
Example 7

Find

\[ \int_{3}^{7} \frac{1}{\sqrt{x - 3}} \, dx. \]