LECTURE OUTLINE
Maxima and Minima

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Math 8
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Review: Critical Points

We also call a point \((a, b)\) a \textit{critical point} of \(f\) if and only if \(\nabla f(a, b) = 0\) or either \(f_x\) or \(f_y\) does not exist.

**Fact:** If \(f\) has a local maximum or minimum at \((a, b)\) and the partial derivatives of \(f\) exist there, then \(\nabla f = 0\).

**Example:** Find the critical points of \(f(x, y) = x^2 + y^2 - 2x - 6y + 14\). Determine whether these points are minima or maxima.

**Example:** Find the critical points of \(f(x, y) = y^2 - x^2\). Determine whether these points are minima or maxima.
Second Derivative Test

Second Derivative Test: Suppose \((a, b)\) is a critical point of \(f(x, y)\) and that the second partial derivatives of \(f\) are continuous on a disk with center \((a, b)\). Let

\[
D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2
\]

(a) If \(D > 0\) and \(f_{xx}(a, b) > 0\), then \(f(a, b)\) is a local minimum.

(a) If \(D > 0\) and \(f_{xx}(a, b) < 0\), then \(f(a, b)\) is a local maximum.

(c) If \(D < 0\), then \(f(a, b)\) is not a local minimum or local maximum. (This is called a saddle point.)
Example 3: Apply the

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

test to our previous examples.

Example 4: Find the critical points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$ Determine whether these points are minima, maxima, or saddles.
Extreme Value Theorem: If $f$ is a continuous on a closed, bounded set $D$ in $\mathbb{R}^2$, then $f$ attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points $(x_1, y_1)$ and $(x_2, y_2)$ in $D$.

To find the absolute maximum and minimum values of a continuous function on a closed, bounded set $D$:
1. Find the value of $f$ at each of the critical points of $f$ in $D$.
2. Find the extreme values of $f$ on the boundary of $D$.
3. The largest of the values from steps 1 and 2 is the absolute maximum while the smallest is the absolute minimum.
Example 5: Find the absolute maximum of 
\[ f(x, y) = x^2 + y^2 + x^2y + 4 \]
of the set 
\[ D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}. \]

Example 2: Find the dimensions of the largest volume rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane 
\[ x + 2y + 3z = 6. \]
The wave equation is

\[ \frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}. \]

Show that \( f(x + at) + g(x - at) \) solves the wave equation.

Suppose you lift a sting in to the position given by \( e^{-x^2} \) and let it go. Explore what happens.