Two contour maps are shown. One is for a function whose graph is a cone. The other is for a function whose graph is a paraboloid. Which is which? Why? I is a paraboloid and II is a cone.

Notice that the circles in II are a constant distance apart and that the ones in I are closer together. This means the height of II is increasing at a constant rate while the height of I is increasing at an increasing rate. Recall that the "slope" along the side of a cone is constant and the "slope" along the side of a paraboloid is changing.

**2.** Draw a contour map of the function showing level curves.

\[ f(x,y) = \frac{y}{x^2+y^2}, \quad \text{let } k = \frac{y}{x^2+y^2} \Rightarrow x^2+y^2 = \frac{y}{k} \]

\[ \Rightarrow x^2+y^2 - \frac{y}{k} = 0 \Rightarrow x^2 + \left( y - \frac{1}{2k} \right)^2 = \frac{1}{4k^2} \]

This is a circle centered at \((0, \frac{1}{2k})\) with radius \(\frac{1}{2k}\). When \(k=0\), \(y=0\) is the x-axis.
Sketch both the contour map and a graph of the function and compare them.

\[ f(x, y) = \sqrt{36 - 9x^2 - 4y^2} \Rightarrow k = \sqrt{36 - 9x^2 - 4y^2} \Rightarrow k^2 = 36 - 9x^2 - 4y^2 \]
\[ 9x^2 + 4y^2 + k^2 = 36 \Rightarrow (\frac{y}{2})^2 + (\frac{x}{3})^2 + (\frac{k}{6})^2 = 1 \]

\[ \Rightarrow \text{a family of ellipses with major axis } = y \text{ axis} \]
\[ k = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0) \]

If we lift each ellipse on the contour map to a height \( z = k \), we have horizontal traces on the graph.

4. Sketch some equipotential curves if \( V(x, y) = \frac{C}{\sqrt{r^2 - x^2 - y^2}} \)

\[ k = \frac{C}{\sqrt{r^2 - x^2 - y^2}} \Rightarrow r^2 - x^2 - y^2 = (\frac{C}{k})^2 \Rightarrow x^2 + y^2 = r^2 - (\frac{C}{k})^2 \]

\[ \Rightarrow \text{a family of circles. Note: as } k \to \infty, \frac{C}{k} \to 0 \rightarrow r^2 - (\frac{C}{k})^2 \to r^2 \]
\[ k - \frac{C}{k} \to x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0) \]
5. Describe the level surfaces of \( f(x, y, z) = x^2 + 3y^2 + 5z^2 \)

\[ k = x^2 + 3y^2 + 5z^2 \Rightarrow \text{family of ellipsoids}, \ k > 0 \]

\[ k = 0 \Rightarrow x^2 + 3y^2 + 5z^2 = 0 \Rightarrow (0, 0, 0) \]

6. Explain why each function is continuous or discontinuous.

(a) The outdoor temperature as a function of longitude, latitude & time.

Since small changes in longitude, latitude, & time produce only small changes (no jumps) in temperature, this function is continuous.

(b) The cost of a taxi ride as a function of distance traveled & time.

Most taxis charge a certain amount per fraction of a mile, plus a flat fare. So the cost increases in jumps every fraction of a mile (or per minute) and so is not continuous.

7 & 8. Find the limit, if it exists, or show that the limit does not exist.

\( \lim_{(x, y) \to (0, 0)} \frac{x^2}{x^2 + y^2} \)

First take the limit along the x-axis.

\( \lim_{(x, 0) \to (0, 0)} \frac{x^2}{x^2 + 0^2} = 1 \). Next, take the limit along the y-axis.

\( \lim_{(0, y) \to (0, 0)} \frac{y^2}{0^2 + y^2} = 0 \). Since 1 ≠ 0, the limit does not exist.

8. \( \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{(x, y) \to (0, 0)} \frac{(x^2 - y^2)(x^2 + y^2)}{(x^2 + y^2)} = \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} = 0 \)
9. Determine the set of points where \( F(x,y) = e^{xy} + \sqrt{x+y^2} \) is discontinuous.

Both \( e^{xy} \) and \( \sqrt{x+y^2} \) are continuous on their respective domain, i.e., \( e^{xy} \) is continuous everywhere on \( \mathbb{R}^2 \) and \( \sqrt{x+y^2} \) is continuous when \( x+y^2 \geq 0 \). So \( F(x,y) \) is continuous when \( x+y^2 \geq 0 \) or \( \{(x,y) \in \mathbb{R}^2 | x \geq -y^2\} \).

10. What are the meanings of the partial derivatives \( \frac{\partial h}{\partial v} \) and \( \frac{\partial h}{\partial t} \)?

\( \frac{\partial h}{\partial v} \) represents the rate of change of \( h \) when we fix \( t \). This describes how quickly the wave heights change when the wind speed changes (at a fixed time). \( \frac{\partial h}{\partial t} \) represents the rate of change of \( h \) when we fix \( v \). This describes how quickly the wave heights change when the time changes (at a fixed wind speed).

6. Estimate the values of \( f_x(40,15) \) and \( f_v(40,15) \). What are the practical interpretations of these values?

\[ f_x(40,15) \approx \lim_{h \to 0} \frac{f(40+h,15) - f(40,15)}{h} \]

Approximate with \( h = 10, -10 \)

\[ f_x(40,15) \approx \frac{f(50,15) - f(40,15)}{10} = \frac{36 - 25}{10} = 1.1 \]

\[ f_x(40,15) \approx \frac{f(30,15) - f(40,15)}{-10} = \frac{16 - 25}{-10} = .9 \]

If we take the average, we have \( f_x(40,15) \approx 1.0 \). This means that when a 40-knot wind has been blowing for 15 hrs, the wave heights should increase by about 1 ft for every knot that the wind increases.

\[ f_v(40,15) = \lim_{h \to 0} \frac{f(40,15+h) - f(40,15)}{h} \]

Approximate with \( h = 5, -5 \)
\[ f_t(40,15) = \frac{f(40,20) - f(40,15)}{5} = \frac{28 - 25}{5} = 0.6 \]
\[ f_t(40,15) = \frac{f(40,10) - f(40,15)}{5} = \frac{21 - 25}{5} = -0.8 \]

If we take the average, we have \( f_t(40,15) \approx 0.7 \). This means that when a 40 knot wind has been blowing for 15 hours, the wave heights should increase by 0.7 feet per extra hour the wind blows.

\[ \text{(a)} \text{ What appears to be the value of } \lim_{t \to \infty} \frac{\partial h}{\partial t} \]

If we fix \( v \) and look at how \( f(v, t) \) changes, we see that it increases less and less as \( t \) increases, becoming nearly constant. So \( \lim_{t \to \infty} \frac{\partial h}{\partial t} = 0 \).

\[ \text{(b)} \text{ Use the contour map to estimate } f_x(2,1) \text{ & } f_y(2,1) \]

To estimate \( f_x(2,1) \), we start at \((2,1)\) and keep \( y = 1 \). \( f(2,1) = 10 \). If we let \( x \) increase, we find \( f(2,1) = 12 \) about 0.6 units to the right of \((2,1)\). So one estimate of \( f_x \) is 2/6. If we let \( x \) decrease, we find \( f(2,1) = 8 \) about 0.9 units to the left of \((2,1)\). So another estimate of \( f_x \) is \(-\frac{2}{9}\).

If we average these, we find \( f_x(2,1) \approx 2.8 \). Similarly, to estimate \( f_y(2,1) \), we start at \((2,1)\) and keep \( x = 1 \). If we let \( y \) increase, we find \( f(2,1) = 8 \) about 0.9 units above \((2,1)\). So one estimate of \( f_y \) is \(-\frac{2}{9}\). If we let \( y \) decrease, we find \( f(3,1) = 12 \) about 1 unit below \((2,1)\). So another estimate for \( f_y \) is \(-\frac{2}{9}\). Averaging these we have \( f_y(2,1) \approx -2.1 \).
Find the first partial derivative of the function,
\[ f(x, y) = x^5 + 3x^3 y^2 + 3xy^4 \]
\[ f_x = 5x^4 + 9x^2 y^2 + 3y^4 \]
\[ f_y = 6x^3 y + 12xy^3 \]

13. \[ f(x, t) = \arctan \left( \frac{x}{\sqrt{t}} \right) \]
\[ f_x = \frac{1}{1 + \left( \frac{x}{\sqrt{t}} \right)^2} \cdot \frac{1}{\sqrt{t}} = \frac{1}{1 + \frac{x^2}{t}} \cdot \frac{1}{\sqrt{t}} = \frac{x}{2 \sqrt{t} (1 + \frac{x^2}{t})} \]
\[ \frac{dy}{dy} \text{ arctan} (u(t)) = \frac{1}{1 + u^2} \]

14. \[ f(x, y, z) = x^2 e^{yz} \]
\[ f_x = 2xe^{yz} \]
\[ f_y = xe^{yz} \]
\[ f_z = x^2 e^{yz} \]

15. Find all the second partial derivatives of \( f(x, y) = \ln (3x + 5y) \)
\[ f_x = \frac{1}{3x + 5y} \cdot 3 = \frac{3}{3x + 5y} \]
\[ f_y = \frac{1}{3x + 5y} \cdot 5 = \frac{5}{3x + 5y} \]
\[ f_{xx} = \frac{\partial}{\partial x} \left( \frac{3}{3x + 5y} \right) = -\frac{9}{(3x + 5y)^2} \]
\[ f_{xy} = \frac{\partial}{\partial y} \left( \frac{3}{3x + 5y} \right) = -\frac{15}{(3x + 5y)^2} \]
\[ f_{yx} = \frac{\partial}{\partial x} \left( \frac{5}{3x + 5y} \right) = -\frac{15}{(3x + 5y)^2} \]
\[ f_{yy} = \frac{\partial}{\partial y} \left( \frac{5}{3x + 5y} \right) = -\frac{25}{(3x + 5y)^2} \]
10. Find the rate of change of temperature with respect to distance at the point \((2,1)\) in the \(x\)-direction and the \(y\)-direction.

\[ T(x,y) = \frac{60}{1+x^2+y^2} \]

\[ T_x = -\frac{60}{(1+x^2+y^2)^2} \cdot 2x \]
\[ T_x(2,1) = \frac{-120(2)}{(1+2^2+1^2)^2} = \frac{-240}{36} = -\frac{20}{3} \]

\[ T_y = -\frac{60}{(1+x^2+y^2)^2} \cdot 2y \]
\[ T_y(2,1) = \frac{-120(1)}{(1+2^2+1^2)^2} = \frac{-120}{36} = -\frac{10}{3} \]

\[ \text{So at } (2,1) \text{ the temperature is decreasing at } \frac{20}{3}^\circ C/m \text{ in the } x\text{-direction and } \frac{10}{3}^\circ C/m \text{ in the } y\text{-direction.} \]