Variance of Discrete Random Variables

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Definition

Let $X$ be a numerically valued random variable with expected value $\mu = E(X)$. Then the variance of $X$, denoted by $V(X)$, is

$$V(X) = E((X - \mu)^2).$$
Standard Deviation

The standard deviation of $X$, denoted by $D(X)$, is $D(X) = \sqrt{V(X)}$. We often write $\sigma$ for $D(X)$ and $\sigma^2$ for $V(X)$. 
Calculation of Variance

**Theorem.** If $X$ is any random variable with $E(X) = \mu$, then

$$V(X) = E(X^2) - \mu^2.$$
Poisson Distribution

- Let $X$ be a Poisson random variable with parameter $\lambda$.
- What is $V(X)$?
Properties of Variance

**Theorem.** If $X$ is any random variable and $c$ is any constant, then

$$V(cX) = c^2 V(X)$$

and

$$V(X + c) = V(X).$$
Theorem. Let $X$ and $Y$ be two independent random variables. Then

$$V(X + Y) = V(X) + V(Y).$$
Example

• Let $X$ be an $n$ Bernoulli trials process.

• What is $V(X)$?
Theorem. Let $X_1, X_2, \ldots, X_n$ be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let

$$S_n = X_1 + X_2 + \cdots + X_n$$

be the sum, and

$$A_n = \frac{S_n}{n}$$

be the average. Then

$$E(S_n) = n\mu,$$
$$V(S_n) = n\sigma^2,$$
$$E(A_n) = \mu,$$
$$V(A_n) = \frac{\sigma^2}{n}.$$
Example

• Let $T$ denote the number of trials until the first success in a Bernoulli trials process.

• What is the variance of $T$?
Continuous Random Variables
Expected Value

Definition. Let $X$ be a real-valued random variable with density function $f(x)$. The expected value $\mu = E(X)$ is defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) \, dx,$$

provided the integral

$$\int_{-\infty}^{+\infty} |x| f(x) \, dx$$

is finite.
Properties

• If $X$ and $Y$ are real-valued random variables and $c$ is any constant, then

$$E(X + Y) = E(X) + E(Y),$$
$$E(cX) = cE(X).$$

• More generally, if $X_1, X_2, \ldots, X_n$ are $n$ real-valued random variables, and $c_1, c_2, \ldots, c_n$ are $n$ constants, then

$$E(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \cdots + c_nE(X_n).$$
Example

• Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5:00 and 6:00 p.m. on Tuesday.

• Suppose each arrives at a time between 5:00 and 6:00 chosen at random with uniform probability.

• Let $Z$ be the random variable which describes the length of time that the first to arrive has to wait for the other.

• What is $E(Z)$?
Expectation of a Function of a Random Variable

**Theorem.** If $X$ is a real-valued random variable and if $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous real-valued function with domain $[a, b]$, then

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x)f_X(x) \, dx,$$

provided the integral exists.
Expectation of the Product of Two Random Variables

**Theorem.** Let $X$ and $Y$ be independent real-valued continuous random variables with finite expected values. Then we have

$$E(XY) = E(X)E(Y).$$
Example

- Let \( Z = (X, Y) \) be a point chosen at random in the unit square.

- What is \( E(X^2Y^2) \)?
Variance

**Definition.** Let $X$ be a real-valued random variable with density function $f(x)$. The variance $\sigma^2 = V(X)$ is defined by

$$\sigma^2 = V(X) = E((X - \mu)^2).$$
Computation

Theorem. If $X$ is a real-valued random variable with $E(X) = \mu$, then

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx .$$
Properties of the variance

• If $X$ is a real-valued random variable defined on $\Omega$ and $c$ is any constant, then

$$V(cX) = c^2 V(X),$$

$$V(X + c) = V(X).$$
• If $X$ is a real-valued random variable with $E(X) = \mu$, then

$$V(X) = E(X^2) - \mu^2.$$
\begin{itemize}
  \item If $X$ and $Y$ are independent real-valued random variables on $\Omega$, then
  \[ V(X + Y) = V(X) + V(Y). \]
\end{itemize}