Math 54 Final Exam

You may use and may only use Armstrong’s text, your class notes, homework solutions which you wrote up, homework solutions placed on our web site, and your instructor for help. Prove all claims. You may use without proof any results proved in chapters 2-5 of the book and any results from assigned homework. Open ended questions should be answered with a proof or a counter example, where any counter example produced should be proved to be a counter example.

Part 1: Basic Concepts

Call a space locally compact if every point is contained in an open set whose closure is compact. Given a locally compact Hausdorff space $X$ which is not compact we may form a new space $\tilde{X}$ which as a set is $X \cup \{\infty\}$ where $\{\infty\}$ is a set containing a single point denoted by $\infty$. Call a set open in $\tilde{X}$ if it is either an open set in $X$, all of $X \cup \{\infty\}$, or any set in the form $(X - K) \cup \{\infty\}$ with $K$ a compact subset of $X$. $\tilde{X}$ is called the one point compactification of $X$.

For this problem let $C$ be the infinite cylinder in $E^3$ described by

$$\{(x, y, z) \in E^3 \mid x^2 + y^2 = 1\},$$

and let $O_4$ be the disjoint union of four copies of the open interval $(0, 1)$.

1. (10 points) Prove the following.

   (a) The open sets described above indeed form a topology on $X \cup \{\infty\}$.
   (b) $\tilde{X}$ is a compact Hausdorff space.
   (c) $X$ is dense in $\tilde{X}$.
   (d) The inclusion map of $X$ into $\tilde{X}$ is an embedding.
   (e) Sketch what $\tilde{C}$ and $\tilde{O}_4$ will look like (by embedding them in $E^3$).

2. (10 points) Let $X$ and $Y$ be locally compact Hausdorff space and prove the following.

   (a) A continuous map $f$ from $X$ to $Y$ extends to a continuous map from $\tilde{X}$ to $\tilde{Y}$ if and only if $f^{-1}(K)$ is compact for every compact set $K$ in $Y$.
   (b) If $X$ and $Y$ are homeomorphic then $\tilde{X}$ and $\tilde{Y}$ are homeomorphic.
   (c) Find two spaces which are not homeomorphic but have homeomorphic one point compactifications.

Part 2: Identification Spaces.

Let $C_4$ be the disjoint union of four copies of $[0, 1]$, and $P_4$ be the partition of $C_4$ into sets which includes the single points in the form $p \in (0, 1)$ for one of the copies of $[0, 1]$ and the set containing all the end points (i.e. $\{0, 1, 0, 1, 0, 1\}$).

Let $M$ be the “paper towel role” in $E^3$ described by

$$\{(x, y, z) \in E^3 \mid x^2 + y^2 = 1, -1 \leq z \leq 1\}.$$
Let $P$ be the partition of this set which includes the singles points $(x, y, z) \in M$ when $|z| \neq 1$ and the set

$$\{(x, y, z) \in M \mid |z| = 1\}.$$ 

Let $N$ be the regular octagon in $E^2$ as pictured below, and let $Q$ be the partition described by identifying points in the boundary as in the figure.

1. (10 points) Prove the following.
   (a) The identification space $I_P$ is homeomorphic to $\tilde{O}_4$.
   (b) The identification space $I_P$ is homeomorphic to $\tilde{C}$.

2. (5 points) Prove that $I_Q$ is a compact surface.

3. (5 points) Prove that a retract is an identification map.

Part 3: The fundamental group and covering spaces.

1. (10 points) Find $\pi_1(S^2 - \{p, q\})$ where $p$ and $q$ are two distinct points in the sphere $S^2$.

2. (10 points) Prove a simply connected space can have only one path connected cover and describe this cover.

3. (10 points) Prove that $\pi_1(\tilde{O}_4)$ is not a commutative group.

4. (10 points) Using only what we know, prove $I_Q$ is not homeomorphic to the sphere $S^2$. 

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