Reference: Today we will apply some of the fundamental group ideas we have been developing. Here I collect the results we will be using and proving.

Some facts about the fundamental group $\pi_1(X,p)$:

1. **A Deck Like Action:** Given a topological space $X$ and a group $G$ given the discrete topology. A continuous action of $G$ on $X$ will be called deck like if for every $p \in X$ there is an open set $U \subset X$ such that $U \cap gU$ is empty for every $g \in G - id$.

   **Example:** The action of $\mathbb{Z}_2$ (which is isomorphic to $\{1, -1\}$ under multiplication) on $S^n = \{x \in \mathbb{E}^{n+1} \mid ||x|| = 1\}$ given by $-1 \cdot x = -x$ is a deck like action.

   **Example:** The action of the integers on $\mathbb{R}$ given by $m \cdot x = x + m$ is a deck like action.

2. **(The Deck Theorem)** If $X$ is simply connected and $G$ acts on $X$ on a deck like way, then $\pi_1(X/G)$ is isomorphic to $G$.

   **Example:** We know that $S^n$ is simply connected for $n > 1$, hence $\pi_1(P^n)$ is $\mathbb{Z}_2$.

   **Example:** We know that $\mathbb{R}$ is simply connected for $n > 1$, hence $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$.

3. **The Isomorphism:** Let the identification map be called $\pi : X \to X/G$. The isomorphism in the deck theorem is given by fixing $p \in X/G$ and $q \in \pi^{-1}(p)$ then defining $\Psi(g) = \langle \pi(\gamma_g) \rangle$ with $\gamma_g$ any path connecting $q$ and $g \cdot q$.

4. **Mapping Properties:** A continuous map $f : X \to Y$ induces a homomorphism between fundamental groups $f_* : \pi_1(X,p) \to \pi_1(Y,f(p))$ given by $f_*(\langle \alpha \rangle) = \langle f(\alpha) \rangle$. Further this correspondence respects compositions, i.e. given a continuous map $g : Y \to Z$ we have that $g_* \circ f_* = (g \circ f)_*$.

   **Example:** Let $f : P^n \to S^1$ be a continuous map. Then for any $\langle \alpha \rangle \in \pi_1(P^n, p)$ we have that $\langle f(\alpha) \rangle = id \in \pi_1(S^1, f(p))$. 


Covering Spaces:

1. **Covering space definition and notation:** We say $\tilde{X}$ covers $X$ if there is a continuous map $\pi: \tilde{X} \to X$ such that for every $p \in X$ there is an open set $V \subset X$ with $p \in V$ such that $\pi^{-1}(V) = \{U_\alpha\}$ with the $U_\alpha$ pairwise disjoint and satisfying the property that for each $\alpha$ we have that $\pi$ restrict to $U_\alpha$ is a homeomorphism. In what follows, $q$ will denote an element in $\pi^{-1}(p)$.

**Example:** For any sphere $S^n$ the mapping $\pi$ sending $x$ to the partition element $\{x, -x\}$ makes $S^n$ into a cover of the projective plane $P^n$. Notice when $n = 1$ that we have a cover the circle with via another circle.

2. **(The path-lifting lemma):** If $\gamma$ is a path beginning at $p$, then there is a unique path $\tilde{\gamma}$ in $\tilde{X}$ which begins at $q$ and satisfies $\pi \circ \tilde{\gamma} = \gamma$.

3. **(The homotopy-lifting lemma):** If $F: I \times I \to X$ is a continuous map such that $F(0, t) = F(1, t) = p$ for all $0 \leq t \leq 1$, then there is a unique continuous map $\tilde{F}: I \times I \to \tilde{X}$ which satisfies $\pi \circ \tilde{F} = F$ and $\tilde{F}(0, t) = q$, for all $0 \leq t \leq 1$.

**Typical Use:** Look at the previous example. Take any path $\tilde{\gamma}$ from $q$ to $-q$ for for some $q \in S^n$ and define $\gamma = \pi \circ \tilde{\gamma}$. Notice that $\pi(q) = \pi(-q) = p \in P^n$ hence $\gamma$ is a path and from the path lifting lemma $\tilde{\gamma}$ is indeed the unique lifting of the path $\gamma$ to $S^n$ starting at $q$. Use the homotopy lifting lemma to show that $< \gamma > \neq id$ in $\pi_1(P^n, p)$.

**Nifty applications.**

1. **(the Borsuk-Ulam Theorem)** Call a continuous mapping $f: S^n \to S^m$ antipode preserving if $\{f(x), f(-x)\} = \{f(x), -f(x)\}$ for every $x \in S^n$. Prove there is no antipode preserving mapping from $S^2$ to $S^1$.

2. **(The Meteorological Theorem)** Prove that for any continuous map $f: S^2 \to \mathbb{R}^2$ that there is an $x \in S^2$ such that $f(x) = f(-x)$.

3. **(The Lusternik-Schirrmann Theorem)** Prove that if $S^2$ by 2 + 1 closed set, then one of these sets contains an antipodal pair $\{x, -x\}$. 

2
Retractions

1. **(Retractions)** If $A$ is subspace of $X$ then a continuous map $g : X \to A$ is called a retraction if $g(a) = a$ for all $a \in A$, and $A$ is called a retract of $X$.

   **Example:** $S^1$ is a retract of $E^2 - \{(0,0)\}$.

2. **(Retraction Lemma)** Suppose $A$ is retract of $X$, then the inclusion mapping of $A$ into $X$ induces a one to one homomorphism from $\pi_1(A,a)$ to $\pi_1(X,a)$.

   **Example:** $S^1$ is not a retract of the closed ball in $E^2$, which we will denote $B$.

3. **(Brouwer Fixed Point Theorem)** Any continuous map of $B$ to itself has a fixed point.

4. **(The "There’s Your Scalp!" Theorem)** For every non-vanishing vector field $V$ on $B$ there is a point where $V$ points directly inward and a point where $V$ points directly outward.
A Rigorous Proof of a Great Theorem:

1. **(A Null Homotopy Lemma)** If a continuous map \( f \) of \( S^1 \) into \( X \) extends to a map of \( B \), into \( X \) then \( f_* \) is trivial.

2. **(The Multiplication Mapping)** \( z^n \) restricted to \( S^1 \) is the multiplication by \( n \) map on \( \pi_1(S^1) \).

3. **(An Extension Lemma)** A continuous map \( f \) of \( S^1 \) into \( X \) extends to a map of \( B \) into \( S^1 \) if and only if \( f \) is homotopic to a point.

4. **(The Fundamental Theorem of Algebra:)** Every polynomial equation \( x^n + c_{n-1}x^{n-1} + \cdots + c_0 \) of degree \( n > 0 \) has at least one root.

Separation Theorems

1. **(Tietze Extension Theorem)** Any real valued continuous function defined on a closed subset of a metric space can be extended to the whole space.

2. **(The path-lifting lemma):** If \( \gamma \) is a path beginning at \( p \), then there is a unique path \( \tilde{\gamma} \) in \( \tilde{X} \) which begins at \( q \) and satisfies \( \pi \circ \tilde{\gamma} = \gamma \).

3. **(A non-separation theorem)** Suppose \( f : I \to \mathbb{E}^2 \) is an embedding. Then \( \mathbb{E}^2 - f(I) \) has one component.

4. **(The Jordan Curve Theorem)** Suppose \( f : S^1 \to \mathbb{E}^2 \) is an embedding. Then \( \mathbb{E}^2 - f(S^1) \) has exactly two components, the bounded and the unbounded.

5. **(Schoenflies Theorem)** Suppose \( f : S^1 \subset \mathbb{E}^2 \to \mathbb{E}^2 \) is an embedding, then \( f \) extends to a homeomorphism of \( \mathbb{E}^2 \) to \( \mathbb{E}^2 \). (In particular the bounded component is homeomorphic to a ball and the unbounded component homeomorphic to a ball minus a point).

6. **(A Fact)** The Jordan curve theorem is true in all dimensions but the Schoenflies theorem is not.