MAJOR FACTS ABOUT RINGS

AXIOMS OF RINGS

A nonempty set \( R \) equipped with two binary operations, addition and multiplication, is called a ring \( \textbf{if} \) for all \( a, b, c \in R \):

1. \( a + b = b + a \);
2. \( (a + b) + c = a + (b + c) \);
3. there is \( 0 \in R \) such that \( a + 0 = a \) for all \( a \in R \) (additive identity);
4. there is \( -a \in R \) such that \( a + (-a) = 0 \);
5. \( a(bc) = (ab)c \);
6. \( a(b + c) = ab + bc \) and \( (b + c)a = ba + ca \).

FACT 1. (Rules of multiplication) Let \( R \) be a ring and \( a, b, c \in R \). Then

1. \( a0 = 0a = 0 \);
2. \( a(-b) = (-a)b = -(ab) \);
3. \( (-a)(-b) = ab \);
4. \( a(b - c) = ab - ac \) and \( (b - c)a = ba - ca \).

Furthermore, if \( R \) and a \textbf{unity} element \( \textbf{1} \), then

5. \( (-1)a = -a \);
6. \( (-1)(-1) = 1 \).

FACT 2. (Uniqueness of the Unity and Inverses)

If a ring has a unity, it is \textbf{unique}. If a ring element has a multiplicative inverse, it is \textbf{unique}.

FACT 3. (Subring Test)

Let \( S \in R \) by a subset of a ring \( R \). Then \( S \) is a subring \( \textbf{if and only if} \) \( S \) is closed under subtraction and multiplication, that is, \( a - b \in S \) and \( ab \in S \) for all \( a, b \in S \).