MAJOR FACTS ABOUT HOMOMORPHISMS

FACT 1. (Properties on Group Elements) Let \( \varphi : G \to \overline{G} \) be a homomorphism. Then
a. \( \varphi \) maps the identity of \( G \) into the identity of \( \overline{G} \), that is, \( \varphi(e) = \overline{e} \);
b. for every \( g \in G \) and every integer \( n \), \( \varphi(g^n) = (\varphi(g))^n \);
c. \( |\varphi(g)| \) divides \( |g| \) for all \( g \in G \) such that \( |g| < \infty \);
d. \( \ker \varphi \) is a (normal) subgroup of \( G \);
e. if \( \varphi(g) = g' \in \overline{G} \), then \( \varphi^{-1}(g') = g \ker \varphi \), where \( \varphi^{-1}(g') = \{ x \in G \mid \varphi(x) = g' \} \).

FACT 2. (Properties of Subgroups) Let \( \varphi : G \to \overline{G} \) be an homomorphism, \( H \leq G \), and \( K \leq \overline{G} \). Then
a. \( \varphi(H) = \{ \varphi(h) \mid h \in H \} \) is a subgroup of \( \overline{G} \);
b. if \( H \) is cyclic, then \( \varphi(H) \) is cyclic;
c. if \( H \) is Abelian, then \( \varphi(H) \) is Abelian;
d. if \( H \) is normal in \( G \), then \( \varphi(H) \) is normal in \( \overline{G} \);
e. if \( |\ker \varphi| = n \), then \( \varphi \) is an \( n \)-to-1 mapping from \( G \) onto \( \varphi(G) \);
f. if \( |H| = n \), then \( |\varphi(H)| \) divides \( n \);
g. \( \varphi^{-1}(K) = \{ k \in G \mid \varphi(k) \in K \} \) is a subgroup of \( G \);
h. if \( K \) is a normal subgroup in \( \overline{G} \), then \( \varphi^{-1}(K) \) is a normal subgroup of \( G \);
i. if \( \varphi \) is onto and \( \ker \varphi = \{ e \} \), then \( \varphi \) is an isomorphism from \( G \) to \( \overline{G} \).

FACT 3. (First Isomorphism Theorem) Let \( \varphi : G \to \overline{G} \) be a homomorphism. Then \( G/\ker \varphi \cong \varphi(G) \) and the isomorphism is given by \( g \ker \varphi \mapsto \varphi(g) \).

FACT 4. (Normal Subgroups are Kernels) Every normal subgroup \( N \) of a group \( G \) is a kernel of the homomorphism from \( G \) to \( G/N \) given by \( g \mapsto gN \).