Gram-Schmidt Orthogonalization Process

Lecture 25

March 7, 2007
Orthogonal Vectors

**Definition**

Let $V$ be an inner product space.

1. Two vectors $x$ and $y$ in $V$ are orthogonal if $\langle x, y \rangle = 0$.
2. A subset $S$ of $V$ is orthogonal if any two distinct vectors in $S$ are orthogonal.
3. A vector $x$ in $V$ is a unit vector if $\|x\| = 1$.
4. A subset $S$ of $V$ is orthonormal if $S$ is orthogonal and consists entirely of unit vectors.
5. A subset $S$ of $V$ is an orthonormal basis for $V$ if it is an ordered basis that is orthonormal.
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5. A subset $S$ of $V$ is an **orthonormal basis** for $V$ if it is an ordered basis that is orthonormal.
Why Study Orthogonal and Orthonormal Sets and Basis?

Fact

If $S$ is an orthogonal subset of $V$ consisting of nonzero vectors, then $S$ is linearly independent.

Any finite dimensional inner product space has an orthonormal basis.

If $\beta = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for $V$, then for any $x \in V$,

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$  

The coefficients $\langle x, v_i \rangle$ are called the Fourier coefficients.

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Theorem

Let $V$ be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of $V$. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$
Mathematical Induction

**Fact**

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Mathematical induction is used to prove that every statement in an infinite sequence of statements is true. It is done by

- proving that the first statement in the infinite sequence of statements is true, and then

- proving that if any one statement in the infinite sequence of statements is true, then so is the next one.
Definition

The simplest and most common form of mathematical induction proves that a statement holds for all natural numbers $n$ and consists of two steps:

1. The basis: showing that the statement holds when $n = 0$.
2. The inductive step: showing that if the statement holds for $n = m$, then the same statement also holds for $n = m + 1$. 

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Mathematical Induction: The First Example

Example

Show that

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}. \]
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for $2 \leq k \leq n$. 

**Theorem**

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Thank you and good luck!
The End!