Math 24 Spring 2006 Assignment 1 Key

Due Friday, April 7.

(1) Is \( \ln x \) (natural logarithm) a linear transformation on the vector space \( \mathbb{R} \)? Justify your answer.

**Answer:** No, it is not. A linear transformation must commute with the vector operations of addition and scalar multiplication, and it is not the case that \( \ln(x+y) = \ln x + \ln y \) or that \( \ln(cx) = c \ln x \).

(2) Define the operation \( * \) on \( \mathbb{R}^>0 \) (positive real numbers) by

\[
 a * b = \sqrt{ab}.
\]

Is \( (\mathbb{R}^>0, *) \) a group? If not, which of the group axioms fails?

**Answer:** \( (\mathbb{R}^>0, *) \) is not a group. There is no identity element, because for a given \( a \) the element \( b \) which gives \( a * b = a \) is \( a \) itself, which means there is no single element which works for all \( a \). Therefore there also cannot be inverses. In fact, this operation is not even associative: the only time \( a * (b * c) = (a * b) * c \) is when \( a = c \).

[all three axiom failures not required; one is sufficient.]

(3) Define the operation \( * \) on \( 2\mathbb{Z} = \{2n : n \in \mathbb{Z}\} \) by

\[
 a * b = a + b.
\]

Is \( (2\mathbb{Z}, *) \) a group? If not, which of the group axioms fails?

**Answer:** \( (2\mathbb{Z}, *) \) is a group.

[unneeded but probably commonly-given information: the operation is closed because the sum of two even numbers is even. It is associative because it is just ordinary addition. The identity is \( 0 = 2 \cdot 0 \) and the inverse of \( 2n \) is \(-2n = 2(-n)\).]

(4) Consider the field \( \{\{0,1,2\} \), + mod 3, \( \cdot \) mod 3\) similar to one discussed in class. Show that for all \( a, b \) in the field,

\[
 (a + b)^3 = a^3 + b^3.
\]

**Answer:** If we expand the binomial, we get

\[
 (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.
\]

Since we are working mod 3, the middle two terms are equal to zero no matter what \( a \) and \( b \) are, so the right hand side simplifies to \( a^3 + b^3 \).