Application of Linear Algebra to Economics

• Wassily Leontief
  – divided U.S. economy into 500 sectors (e.g. coal industry, automotive industry, communications)
  – for each sector, wrote linear equation describing how sector distributes output to other sectors

• Leontief “input-output” (or “production”) model

Terminology

• $n$: number of sectors in nation’s economy
• $x \in \mathbb{R}^n$ production vector: output of each sector for year
• $d \in \mathbb{R}^n$ final demand vector: value of goods and services demanded from sectors by non-productive part of economy
• intermediate demand: inputs producers need for production

Leontief’s question: is there a production level such that the total amount produced equals the total demand for production?

Is there an $x \in \mathbb{R}^n$ such that $x = \text{intermediate demand} + d$?
The Model

- hold prices of goods and services constant
- measure unit of input and output in millions of dollars
- basic assumption: for each sector, there is a unit consumption vector \( \mathbf{c} \) listing inputs needed per unit of output of sector

Example \((n = 3)\)

<table>
<thead>
<tr>
<th>Purchased from</th>
<th>Inputs Consumed per Unit of Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Manufacturing</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>0.50</td>
</tr>
<tr>
<td>Agriculture</td>
<td>0.20</td>
</tr>
<tr>
<td>Services</td>
<td>0.10</td>
</tr>
</tbody>
</table>

\[ \uparrow \] \[ \uparrow \] \[ \uparrow \]

\( \mathbf{c}_1 \) \( \mathbf{c}_2 \) \( \mathbf{c}_3 \)

What will the manufacturing sector consume if it produces 100 units?

50 units from manufacturing, 20 units from agriculture, 10 units from services
Suppose sector

- has unit consumption vector \( \mathbf{c} \)
- produces \( x \) units of output

What is sector’s intermediate demand? \( x\mathbf{c} \)

total intermediate demand = \( x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = Cx \), where \( C \) is consumption matrix \( C = [\mathbf{c}_1 \cdots \mathbf{c}_n] \)

Leontief’s question: is there an \( \mathbf{x} \in \mathbb{R}^n \) such that \( \mathbf{x} = C\mathbf{x} + \mathbf{d} \)? Alternatively, is there an \( \mathbf{x} \in \mathbb{R}^n \) such that \( (I_n - C)\mathbf{x} = \mathbf{d} \)?

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**Example \((n = 3)\)**

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<tr>
<th>Purchased from</th>
<th>Manufacturing</th>
<th>Agriculture</th>
<th>Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturing</td>
<td>0.50</td>
<td>0.40</td>
<td>0.20</td>
</tr>
<tr>
<td>Agriculture</td>
<td>0.20</td>
<td>0.30</td>
<td>0.10</td>
</tr>
<tr>
<td>Services</td>
<td>0.10</td>
<td>0.10</td>
<td>0.30</td>
</tr>
</tbody>
</table>

\[
\mathbf{C} = \begin{bmatrix}
0.50 & 0.40 & 0.20 \\
0.20 & 0.30 & 0.10 \\
0.10 & 0.10 & 0.30
\end{bmatrix}
\]
Example \((n = 3)\)

Suppose the final demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services. What is the production level that will satisfy this demand?

\[
I_3 - C = \begin{bmatrix}
0.50 & -0.40 & -0.20 \\
-0.20 & 0.70 & -0.10 \\
-0.10 & -0.10 & 0.70
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.50 & -0.40 & -0.20 & 50 \\
-0.20 & 0.70 & -0.10 & 30 \\
-0.10 & -0.10 & 0.70 & 20
\end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 225.9 \\
0 & 1 & 0 & 118.5 \\
0 & 0 & 1 & 77.8
\end{bmatrix}
\]

- \(I_n - C\) invertible implies \(x = (I_n - C)^{-1}d\)
- in most practical cases, \(I_n - C\) is invertible

**column sum:** sum of entries in column

**Theorem:** Let \(C\) be the consumption matrix for an economy and \(d\) the final demand vector. If \(C\) and \(d\) have non-negative entries and if each column sum of \(C\) is less than 1, then \(I - C\) is invertible, and the production vector

\[
x = (I - C)^{-1}d
\]

has non-negative entries and is the unique solution of

\[
x = Cx + d.
\]
Note: sector should need less than one unit’s worth of inputs to produce one unit of output, so column sums of consumption matrix should all be less than 1

- suppose $d$ is presented to various sectors at start of year and sectors set $x = d$
- intermediate demand $= Cd$
- to meet demand of $Cd$, sectors need inputs of $C(Cd) = C^2d$, creating second round of intermediate demand of $C(C^2d) = C^3d$
- theoretically, process continues indefinitely

<table>
<thead>
<tr>
<th>Demand</th>
<th>Inputs Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Demand</td>
<td>$d$</td>
</tr>
<tr>
<td>Intermediate demand</td>
<td>$Cd$</td>
</tr>
<tr>
<td>round 1</td>
<td>$Cd$</td>
</tr>
<tr>
<td></td>
<td>$C(Cd) = C^2d$</td>
</tr>
<tr>
<td>round 2</td>
<td>$C^2d$</td>
</tr>
<tr>
<td></td>
<td>$C(C^2d) = C^3d$</td>
</tr>
<tr>
<td>round 3</td>
<td>$C^3d$</td>
</tr>
<tr>
<td></td>
<td>$C(C^3d) = C^4d$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$x = d + Cd + C^2d + C^3d + \cdots$

$= (I_n + C + C^2 + C^3 + \cdots)d$
• \((I_n - C)(I_n + C + C^2 + \cdots + C^m) = I_n - C^{m+1}\)

• if all column sums of \(C\) are less than 1, then
  - \(I_n - C\) is invertible
  - \(C^m \to 0\) as \(m \to \infty\)
  - \(I_n - C_0 \to I_n\) as \(m \to \infty\) (idea: \(0 < t < 1\) implies \(t^m \to 0\) as \(m \to \infty\))

• \((I_n - C)^{-1} \approx I_n + C + C^2 + \cdots + C^m\); i.e., right-hand side can be made as close to \((I_n - C)^{-1}\) as we want by taking \(m\) large enough

• in actual input-output models, powers of consumption matrix \(C\) approach 0 quickly, and for given final demand \(d\), vectors \(C^m d\) approach 0 quickly

• entries in \((I_n - C)^{-1}\) can be used to predict how production \(x\) will have to change when \(d\) changes: entries in column \(j\) of \((I_n - C)^{-1}\) are increased amounts various sectors will have to produce to satisfy increase of one unit in final demand for output from sector \(j\)