Discrete Probabilities

Summer 2006
Random Variables and Sample Spaces

- We represent the outcome of the experiment by a capital Roman letter, such as $X$, called a *random variable*.

- The *sample space* of the experiment is the set of all possible outcomes. If the sample space is either finite or countably infinite, the random variable is said to be *discrete*.

- The elements of a sample space are called outcomes.

- A subset of the sample space is called an *event*.
Let $X$ be a random variable which denotes the value of the outcome of a certain experiment, and assume that this experiment has only finitely many possible outcomes. Let $\Omega$ be the sample space of the experiment (i.e., the set of all possible values of $X$, or equivalently, the set of all possible outcomes of the experiment.) A distribution function for $X$ is a real-valued function $m$ whose domain is $\Omega$ and which satisfies:

1. $m(\omega) \geq 0$, for all $\omega \in \Omega$, and

2. $\sum_{\omega \in \Omega} m(\omega) = 1$. 
For any subset $E$ of $\Omega$, we define the probability of $E$ to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m(\omega).$$
Examples

Three people, A, B, and C, are running for the same office, and we assume that one and only one of them wins. Suppose that A and B have the same chance of winning, but that C has only 1/2 the chance of A or B. What is the probability to win for each of the three people?
Basic Set Operations

• Then the union of $A$ and $B$ is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

• The intersection of $A$ and $B$ is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$
• The difference of $A$ and $B$ is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$ 

• The complement of $A$ is the set

$$\tilde{A} = \{x \mid x \in \Omega \text{ and } x \notin A\}.$$
Properties

The probabilities assigned to events by a distribution function on a sample space $\Omega$ satisfy the following properties:

1. $P(E) \geq 0$ for every $E \subset \Omega$.

2. $P(\Omega) = 1$.

3. If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.

4. If $A$ and $B$ are disjoint subsets of $\Omega$, then $P(A \cup B) = P(A) + P(B)$.

5. $P(\tilde{A}) = 1 - P(A)$ for every $A \subset \Omega$. 

Properties ...

• For any two events $A$ and $B$,

$$P(A) = P(A \cap B) + P(A \cap \tilde{B}).$$

• If $A$ and $B$ are subsets of $\Omega$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
Uniform Distribution

The *uniform distribution* on a sample space $\Omega$ containing $n$ elements is the function $m$ defined by

$$m(\omega) = \frac{1}{n},$$

for every $\omega \in \Omega$. 
Example

Consider the experiment that consists of rolling a pair of dice. We take as the sample space $\Omega$ the set of all ordered pairs $(i, j)$ of integers with $1 \leq i \leq 6$ and $1 \leq j \leq 6$. Thus,

$$ \Omega = \{(i, j) : 1 \leq i, j \leq 6\} . $$
Odds

If $P(E) = p$, the odds in favor of the event $E$ occurring are $r : s$ ($r$ to $s$) where $r/s = p/(1 - p)$. If $r$ and $s$ are given, then $p$ can be found by using the equation $p = r/(r + s)$. 
Infinite Sample Space

If

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$$

is a countably infinite sample space, then a distribution function is defined exactly as before, except that the sum must be *convergent*. 
Examples

A coin is tossed until the first time that a head turns up. Let the outcome of the experiment, $\omega$, be the first time that a head turns up. Then the possible outcomes of our experiment are

$$\Omega = \{1, 2, 3, \ldots\}.$$  

What is the probability that the coin eventually turns up heads.