Central Limit Theorem
Bernoulli Trials

11/01/2005
Continuous Probability Densities

- Let us construct a spinner, which consists of a circle of unit circumference and a pointer.

- The experiment consists of spinning the pointer and recording the label of the point at the tip of the pointer.
• We let the random variable \( X \) denote the value of this outcome.

• The sample space is clearly the interval \([0, 1)\).

• It is necessary to assign the probability 0 to each outcome.

• The probability

\[
P(0 \leq X \leq 1)
\]

should be equal to 1.
• We would like the equation

\[ P(c \leq X < d) = d - c \]

to be true for every choice of \( c \) and \( d \).

• If we let \( E = [c, d] \), then we can write the above formula in the form

\[ P(E) = \int_{E} f(x) \, dx , \]

where \( f(x) \) is the constant function with value 1.
Density Functions of Continuous Random Variables

Let $X$ be a continuous real-valued random variable. A density function for $X$ is a real-valued function $f$ which satisfies

$$P(a \leq X \leq b) = \int_{a}^{b} f(x) \, dx$$

for all $a, \ b \in \mathbb{R}$. 
• It is not the case that all continuous real-valued random variables possess density functions.

• In terms of the density $f(x)$, if $E$ is a subset of $\mathbb{R}$, then

\[ P(X \in E) = \int_E f(x) \, dx. \]
Example

- In the spinner experiment, we choose for our set of outcomes the interval $0 \leq x < 1$, and for our density function

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- If $E$ is the event that the head of the spinner falls in the upper half of the circle, then $E = \{ x : 0 \leq x \leq 1/2 \}$, and so

$$P(E) = \int_{0}^{1/2} 1 \, dx = \frac{1}{2}.$$
• More generally, if $E$ is the event that the head falls in the interval $[a, b]$, then

$$P(E) = \int_a^b 1\, dx = b - a.$$
Example: Continuous Uniform Density

- The simplest density function corresponds to the random variable $U$ whose value represents the outcome of the experiment consisting of choosing a real number at random from the interval $[a, b]$.

$$f(w) = \begin{cases} 
1/(b-a), & \text{if } a \leq \omega \leq b \\
0, & \text{otherwise.}
\end{cases}$$
Normal Density

• The *normal density* function with parameters $\mu$ and $\sigma$ is defined as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$  

• The parameter $\mu$ represents the "center" of the density.

• The parameter $\sigma$ is a measure of the "spread" of the density, and thus it is assumed to be positive.
Central Limit Theorem for Bernoulli Trials

- We deal only with the case that $\mu = 0$ and $\sigma = 1$.

- We will call this particular normal density function the *standard* normal density, and we will denote it by $\phi(x)$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$
• Consider a Bernoulli trials process with probability $p$ for success on each trial.

• Let $X_i = 1$ or 0 according as the $i$th outcome is a success or failure, and let $S_n = X_1 + X_2 + \cdots + X_n$.

• Then $S_n$ is the number of successes in $n$ trials.

• We know that $S_n$ has as its distribution the binomial probabilities $b(n, p, j)$. 
Standardized Sums

• We can prevent the drifting of these spike graphs by subtracting the expected number of successes $np$ from $S_n$.

• We obtain the new random variable $S_n - np$.

• Now the maximum values of the distributions will always be near 0.

• To prevent the spreading of these spike graphs, we can normalize $S_n - np$ to have variance 1 by dividing by its standard deviation $\sqrt{npq}$.
Definition

The *standardized sum* of $S_n$ is given by

$$S_n^* = \frac{S_n - np}{\sqrt{npq}}.$$

$S_n^*$ always has expected value 0 and variance 1.
• We plot a spike graph with the spikes placed at the possible values of $S_n^*$: $x_0, x_1, \ldots, x_n$, where

$$x_j = \frac{j - np}{\sqrt{npq}}.$$

• We make the height of the spike at $x_j$ equal to the distribution value $b(n, p, j)$. 
• We plot a spike graph with the spikes placed at the possible values of $S_n^*$: $x_0, x_1, \ldots, x_n$, where

$$x_j = \frac{j - np}{\sqrt{npq}}.$$  

• We make the height of the spike at $x_j$ equal to the distribution value $b(n, p, j)$. 

![Graph showing spike heights at possible values of $S_n^*$]
• Let $\varepsilon$ be the distance between consecutive spikes.

• to change the spike graph so that the area under this curve has value 1, we need only multiply the heights of the spikes by $1/\varepsilon$.

• We see that

$$\varepsilon = \frac{1}{\sqrt{npq}}.$$
• Let us fix a value $x$ on the $x$-axis and let $n$ be a fixed positive integer.

• Then the point $x_j$ that is closest to $x$ has a subscript $j$ given by the formula

$$j = \langle np + x \sqrt{npq} \rangle .$$

• Thus the height of the spike above $x_j$ will be

$$\sqrt{npq} b(n, p, j) = \sqrt{npq} b(n, p, \langle np + x_j \sqrt{npq} \rangle) .$$
Central Limit Theorem for Binomial Distributions

**Theorem.** For the binomial distribution $b(n, p, j)$ we have

$$\lim_{n \to \infty} \sqrt{npq} b(n, p, (np + x \sqrt{npq})) = \phi(x),$$

where $\phi(x)$ is the standard normal density.
Approximating Binomial Distributions

- To find an approximation for $b(n, p, j)$, we set
  \[ j = np + x\sqrt{npq} \]

- Solve for $x$
  \[ x = \frac{j - np}{\sqrt{npq}}. \]

- \[ b(n, p, j) \approx \frac{\phi(x)}{\sqrt{npq}} \]
  \[ = \frac{1}{\sqrt{npq}} \phi \left( \frac{j - np}{\sqrt{npq}} \right). \]
Example

- Let us estimate the probability of exactly 55 heads in 100 tosses of a coin.

- For this case \( np = 100 \cdot 1/2 = 50 \) and \( \sqrt{npq} = \sqrt{100 \cdot 1/2 \cdot 1/2} = 5 \).

- Thus \( x_{55} = (55 - 50)/5 = 1 \) and

\[
P(S_{100} = 55) \sim \frac{\phi(1)}{5} = \frac{1}{5} \left( \frac{1}{\sqrt{2\pi}} e^{-1/2} \right) = .0484.
\]