LECTURE OUTLINE

The Dot Product

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Math 15

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Today

- Polar Coordinates
- Projection
- Dot Product
- Orthonormal Basis
- Changing Coordinates
- Polar Coordinate Differentiation
Cylindrical Coordinates

We define cylindrical coordinates via

\[(r, \theta, z)_P = (r \cos(\theta), r \sin(\theta), z).\]

We can find a cylindrical coordinate determining \((x, y, z)\) via

\[(x, y, z) = \left(\sqrt{x^2 + y^2}, \arctan \left(\frac{y}{x}\right), z\right)_P,
\]

for some \(\arctan\). Restricting to \(z = 0\) we have polar coordinates.
When thinking in terms of polar coordinates, we use $\mathbf{r}$ to describe position

$$\mathbf{r} = r\mathbf{\hat{r}}(\theta) = r(\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}),$$

and use $\mathbf{r}$’s perpendicular companion

$$\mathbf{\hat{r}} = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$$

to describe vectors at $(r, \theta)_P$. 

\textit{Polar Coordinates: Vectors}
Circular Motion

An object is moving around a circle of radius 1/2 in the $x,y$-plane (where the units of distance are meters) in a counter clockwise direction at a constant speed of 3 meters per second. Its initial position vector is $(1/2) \hat{i}$.

(a) Describe its position after 6 seconds in polar and Cartesian coordinates.
(b) In both Cartesian and polar coordinates, find a vector representing its velocity when it is located at the point with position vector $(\frac{1}{4}) \hat{i} + (\frac{\sqrt{3}}{4}) \hat{j}$. 
Given two unit vectors $\hat{u}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$ and $\hat{u}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$ and letting $\theta$ be the angle between them we have

$$\cos(\theta) = 1 - 2 \sin\left(\frac{\theta}{2}\right)^2 =$$

$$1 - 2\left|\frac{\hat{u}_2 - \hat{u}_1}{2}\right|^2 = (x_1 x_2 + y_1 y_2 + z_1 z_2)$$
Dot Product

Hence for any \( \vec{v} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \) and \( \vec{w} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \), if we let

\[
\vec{v} \cdot \vec{w} = x_1 x_2 + y_1 y_2 + z_1 z_2
\]

then we have

\[
\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| ||\vec{w}||}
\]

where \( \theta \) is the angle between \( \vec{v} \) and \( \vec{w} \).
We Used...

Lemma:

$$(c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w}).$$