LECTURE OUTLINE
Work and Line Integrals

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Math 8

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Goals

Introduce: Work
The Line Integral
Derivatives and the Dot Product

Theorem:

\[
\frac{d}{dt} (\vec{w}_1 \cdot \vec{w}_2) = \frac{d\vec{w}_1}{dt} \cdot \vec{w}_2 + \vec{w}_1 \cdot \frac{d\vec{w}_2}{dt}
\]

Recall, some notation from last time

\[\vec{w}_1 = x_1(t)\hat{i} + y_1(t)\hat{j} + z_1(t)\hat{k}\]
\[\vec{w}_2 = x_2(t)\hat{i} + y_2(t)\hat{j} + z_2(t)\hat{k}\].
One Consequence (of many)

Theorem: If the curvature is not 0, then

\[ \hat{N} \cdot \hat{T} = 0. \]
Forces

We must distinguish between the total force

\[ \vec{F}_T = m\vec{a} \]

acting on an object and a force \( \vec{F}_i \) acting on an object where

\[ \vec{F}_T = \sum_{i=1}^{n} \vec{F}_i. \]
Suppose a particle is acted on by a force

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

while following the path \((t^2, t^3, t)\) for \(t\) in \([0, 1]\). Can \(\vec{F}\) be the only force acting on the particle?
**Work**

Let \( \vec{F} \) be a force and let \( \gamma \) denote the path determined by \( \vec{r}(t) \) for \( t \) in the interval \([a, b]\). We say the work done by \( \vec{F} \) as an object traverses \( \gamma \) is given by the following *line integral*

\[
W_{\vec{F}}(\gamma) = \int_{\gamma} \vec{F} \cdot d\vec{r} = \int_{a}^{b} \vec{F} \cdot \frac{d\vec{r}}{dt} dt.
\]

(From the first integral \( W_{\vec{F}}(\gamma) \) is independent of the parameterization. Yet, from second integral we see work is most easily computed from a given parameterization.)
Example 1(a)

Let

\[ \vec{F} = x\hat{i} + y\hat{j} + z\hat{k} \]

and \( \gamma \) denote the path determined by

\[ \vec{r}(t) = t^2\hat{i} + t^3\hat{j} + t\hat{k} \]

for \( t \) in the interval \([0, 1]\). Compute the work done by \( \vec{F} \) as our object traverses \( \gamma \).
Potential Energy

For each component force, we define the potential energy associated to this force at each time $t$ to be

$$U_i(\gamma(t)) = -W_{F_i} (\gamma([a, t])).$$

(This notation is sly. It suggests that potential energy should depend only on $\gamma$’s end points. This is not always true, though this is indeed often the case for potential energies of interest to us.)
Example 1(b)

Let

\[ \vec{F} = xi + yj + zk \]

and let \( \gamma \) denote the path determined by

\[ \vec{r}(t) = t^2i + t^3j + tk \]

for \( t \) in the interval \([0, 1]\). Compute the potential energy at \((1, 1, 1)\).
Example 1(c)

Let

\[ \vec{F} = xi + yj + zk \]

and let \( \gamma \) denote your favorite path determined from \((0, 0, 0)\) to \((1, 1, 1)\). Compute the potential energy at \((1, 1, 1)\).