LECTURE OUTLINE
Tangent Planes and Gradients

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Math 15

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Goals

The Chain Rule
The Gradient
The Tangent Plane
**Chain Rule**

Recall

\[
\frac{df(x)}{dt} = \frac{df}{dx} \frac{dx}{dt},
\]

this chain rule generalizes, and we have

\[
\frac{df(x, y)}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

**Ex.** Let \( f(x, y) = x^2 y + y^3 \), \( x(t) = \sin(t) \), and \( y(t) = e^t \). Find \( \frac{d}{dt} (e^t (\sin(t))^2 + e^{3t}) \) in the old way and using the chain rule.
Contour Directions and Mountain Climbs

Pick a direction \( \hat{u} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} \) in the contour plot at the point \((x_0, y_0)\). Travel in this direction via
\[(x_0 + t \cos(\theta))\hat{i} + (y_0 + t \sin(\theta))\hat{j} \] in the contour map. What is our velocity vector on the mountain side at \((x_0, y_0, f(x_0, y_0))\)?

**Answer:** Letting \( \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \), we have our velocity vector on the mountain side is given by
\[\vec{u} + (\nabla f \cdot \vec{u}) \hat{k}.\]
Example

Let \( f(x, y) = \cos(xy)e^{\frac{-x^2 - y^2}{10}} \). Find the gradient of \( f(x, y) \). In each contour direction, what is our velocity vector at \((1, 0, e^{-\frac{1}{10}})\). What is the rate of change of our height in each of these directions?
Notice: the fastest rate of height gain is achieved in the direction $\frac{\nabla f}{|\nabla f|}$, the fastest rate of height loss is achieved in the direction $-\frac{\nabla f}{|\nabla f|}$, while there is no change in height in a direction perpendicular to $\frac{\nabla f}{|\nabla f|}$.
A plane thought the origin is the collection of all position vectors \( \vec{r} \) perpendicular to a fixed vector \( \vec{n} \), the plane’s normal vector. Hence the equation \( \vec{n} \cdot \vec{r} = 0 \) determines such a plane.

While the plane parallel to this plane that contains the point \( \vec{p} \) is determined by the equation \( \vec{n} \cdot (\vec{r} - \vec{p}) = 0 \).

Notice, a plane containing the vectors \( \vec{v}_1 \) and \( s \vec{v}_2 \) has \( \vec{n} = \vec{v}_1 \times \vec{v}_2 \).
The tangent plane at a point on a mountain side is the plane containing the velocity vectors of the curves going through that point.

**Argue that:** the tangent plane at \((x_0, y_0, f(x_0, y_0))\) has normal given by \(-\frac{\partial f}{\partial y} \hat{i} - \frac{\partial f}{\partial x} \hat{j} + \hat{k}\).

Find an equation for the tangent plane of \(f(x, y) = \cos(xy)e^{\frac{-x^2-y^2}{10}}\) at \((1, 0, e^{-\frac{1}{10}})\).