\[ (\hat{u} \times \hat{v}) \cdot \hat{w} = (\hat{u} \times \hat{v}) \cdot \hat{w} = 0 \]  
\[ (5.4) \]

\( \hat{u} \times \hat{v} \) is perpendicular to both \( \hat{u} \) and \( \hat{v} \) so both dot products must be 0.

\[ (a \hat{u}) \times \hat{w} = a (\hat{u} \times \hat{w}) = \hat{u} \times (a \hat{w}) \]  
\[ (5.7) \]

\[ |(a \hat{u}) \times \hat{w}| = |a| |\hat{u}| |\hat{w}| \sin \theta, \]  where \( \theta \) is the angle between \( \hat{u} \) and \( \hat{w} \)

\[ \Rightarrow |(a \hat{u}) \times \hat{w}| = |a| |\hat{u}| |\hat{w}| \sin \theta = |a| |\hat{u} \times \hat{w}| \]

\[ = |(a \hat{u}) \times \hat{w}| = |\hat{u}||a\hat{w}||\sin \theta = |\hat{u} \times (a\hat{w})| \]

So the expressions in (5.7) have equal magnitudes, but what if \( a < 0 \)?

Check for \( a = -1 \): \( (-\hat{u}) \times \hat{w} = -(\hat{u} \times \hat{w}) \) by the right hand rule.

\[ \Rightarrow (a \hat{u}) \times \hat{w} = -a (\hat{u} \times \hat{w}) = \hat{u} \times (a \hat{w}) \]

\[ 133 \quad \hat{u} \cdot (\hat{w} \times \hat{v}) = \hat{u} \cdot [-(\hat{v} \times \hat{w})] = -\hat{u} \cdot (\hat{v} \times \hat{w}) \]

\[ 134 \quad \hat{i} \times (\hat{j} \times \hat{k}) = \hat{i} \times (\hat{k}) = -\hat{j} \]

\[ (\hat{i} \times \hat{i}) \times \hat{j} = (\hat{0}) \times \hat{j} = \hat{0} \]

Since \( -\hat{j} = \hat{0} \), the cross product is not associative.
\[ \omega = (\cos \theta, \sin \theta, 0) \implies \frac{d\omega}{d\theta} = (-\sin \theta, \cos \theta, 0) \]

\[ \frac{d}{d\theta} (\hat{u} \times \hat{i}) = \frac{d\hat{u}}{d\theta} \times \hat{i} + \hat{u} \times \frac{d\hat{i}}{d\theta} \text{, but } \frac{d\hat{i}}{d\theta} = \hat{0} \]

So, \[ \frac{d}{d\theta} (\hat{u} \times \hat{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \hat{i} - 0 \hat{j} + (-\cos \theta) \hat{k} = -\cos \theta \hat{k} \]

Volume = \[ \vec{v} \cdot (\hat{u} \times \hat{w}) = \vec{v} \cdot [-\left( (\hat{u} \times \hat{w}) \right) ] = -\vec{v} \cdot (\hat{u} \times \hat{w}) \]

If \( \vec{v} \) is in the same plane as \( \hat{u} \) and \( \hat{w} \) then their triple product is zero because the create a "flat" parallelipiped with zero volume.

Equivalently, \( \hat{u} \times \hat{w} \) is \( \perp \) to the plane containing \( \hat{u}, \vec{v}, \hat{w} \)

\[ \Rightarrow \vec{v} \cdot (\hat{u} \times \hat{w}) = 0 \text{ since the vectors are } \perp \]
Ordering    Triple Product    Orientation
\[ \hat{i}, \hat{j}, \hat{k} \]    \( \hat{i} \cdot (\hat{j} \times \hat{k}) = +1 \)    positive
\[ \hat{k}, \hat{i}, \hat{j} \]    \( \hat{k} \cdot (\hat{i} \times \hat{j}) = +1 \)    positive
\[ \hat{j}, \hat{k}, \hat{i} \]    \( \hat{j} \cdot (\hat{k} \times \hat{i}) = +1 \)    positive
\[ \hat{j}, \hat{i}, \hat{k} \]    \( \hat{j} \cdot (\hat{i} \times \hat{k}) = -1 \)    negative
\[ \hat{k}, \hat{j}, \hat{i} \]    \( \hat{k} \cdot (\hat{j} \times \hat{i}) = -1 \)    negative
\[ \hat{i}, \hat{k}, \hat{j} \]    \( \hat{i} \cdot (\hat{k} \times \hat{j}) = -1 \)    negative

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\[ 2x + 3y - z = 0 \implies (2, 3, -1) \cdot (x, y, z) = 0 \]

Since every vector \((x, y, z)\) that satisfies this equation is in the plane, the vector \((2, 3, -1)\) must be \perp to the plane.

\[ \implies \vec{r}(t) = (0, 0, 0) + t(2, 3, -1) \] is the parametrized line through the origin \perp to the plane.
\( \vec{v} = (1, 1, 2) \quad \vec{u} = (1, 2, 1) \quad \vec{w} = (2, 1, 1) \)

\[
Volume = \vec{v} \cdot (\vec{u} \times \vec{w}) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (2-1) \cdot 1 - (1-2) \cdot 1 + (1-4) \cdot 2 = -6
\]

The vectors in the order given are negatively oriented.

\( l_1 \) is in the direction \( \vec{a}_1 = (2, -2, 0) - (1, 2, 1) = (1, -4, -1) \)

\( l_2 \) is in the direction \( \vec{a}_2 = (-1, 4, -2) - (2, 1, 5) = (-3, 3, -7) \)

\[
\vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -3 & 3 & -7 \end{vmatrix} = (28+3) \hat{i} - (-7-3) \hat{j} + (3-12) \hat{k} = 31 \hat{i} + 10 \hat{j} - 9 \hat{k}
\]

\( \Rightarrow 31 \hat{i} + 10 \hat{j} - 9 \hat{k} \) is \( \perp \) to \( l_1 \) and \( l_2 \).

\( \vec{u} = (2, 1, 4) \quad \vec{v} = (-1, 3, 8) \)

\[
\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ -1 & 3 & 8 \end{vmatrix} = (8-12) \hat{i} - (16+4) \hat{j} + (4+1) \hat{k} = -4 \hat{i} - 20 \hat{j} + 7 \hat{k}
\]

\[
Area = |\vec{u} \times \vec{v}| = \sqrt{(-4)^2 + (-20)^2 + (7)^2} = \sqrt{465}
\]
\[ \overrightarrow{d_1} = (3, 2, 5) - (1, 1, 1) = (2, 1, 4) \]

\[ \overrightarrow{d_2} = (0, 4, 9) - (1, 1, 1) = (-1, 3, 8) \]

\[ \text{Area}_\triangle = \frac{1}{2} \text{Area}_\square = \frac{1}{2} | \overrightarrow{d_1} \times \overrightarrow{d_2} | = \frac{1}{2} \sqrt{465} \quad \text{from 1st part} \]