LECTURE OUTLINE
Kinematics of Rotation

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Math 15

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Goals

Cross Product and the Determinant

Rigid Body Kinematics
Last time we explored the Cross Product

Nice and Linear:

\[(\vec{v}_1 + c\vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + c\vec{v}_2 \times \vec{w}\]

but, Not Commutative!

\[\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}\]

and Not Associative!

\[(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u})\]

In particular, \(\vec{u} \times \vec{v} \times \vec{w}\) has no good meaning!
Computing a $3 \times 3$ Determinant

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
  u_y & u_z \\
  v_y & v_z \\
\end{vmatrix} \hat{i} - \\
\begin{vmatrix}
  u_x & u_z \\
  v_x & v_z \\
\end{vmatrix} \hat{j} + \\
\begin{vmatrix}
  u_x & u_y \\
  v_x & v_y \\
\end{vmatrix} \hat{k}
\]

\[
\vec{w} \cdot (\vec{u} \times \vec{v}) \equiv \begin{vmatrix}
  w_x & w_y & w_z \\
  u_x & u_y & u_z \\
  v_x & v_y & v_z \\
\end{vmatrix} = \vec{w} \cdot \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  u_x & u_y & u_z \\
  v_x & v_y & v_z \\
\end{vmatrix}
\]

Let $\vec{u} = \hat{i} - 2\hat{k}$, $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{w} = 3\hat{i} + 2\hat{j}$, and compute $\vec{w} \cdot (\vec{u} \times \vec{v})$.
Application to spacial reasoning

Ex: Note we can think of a line as $t\vec{v} + \vec{p}$. Define what it means for two lines to be non-parallel, and find a formula for the distance between two non-parallel lines.

Find the distance between the lines $(\hat{i} - 2\hat{k})t + \hat{i}$ and $\vec{v} = t(\hat{i} + \hat{j} + \hat{k})$. 
Kinematics

For our rotation use $\hat{i}$, $\hat{j}$, and $\hat{k}$ with $\hat{k}$ the axis of rotation and $\hat{i}$ to $\hat{j}$ in the direction of rotation (right hand rule). Give $\hat{r}$ and $\hat{\theta}$ their usual meanings with respect to $\hat{i}$, $\hat{j}$, $\hat{k}$. There is a second "view" we might take to think about the center (center of mass) which we might call $\hat{i}_c$, $\hat{j}_c$, $\hat{k}_c$. A particle moving about another particle can be described by:

$$\vec{r} = \vec{c} + r\hat{r} + z\hat{k}.$$
Usual Book Assumptions:
(1) \( \dot{r} = 0 \) (Not Nutty)
(2) \( z = 0 \) (Not so Nutty)
(3) \( \frac{d\hat{k}}{dt} = 0 \) (Nutty)

Note: (3) The Great Yo-Yo Restriction allows us to assume \( \frac{d\hat{i}}{dt} = 0 \) and \( \frac{d\hat{j}}{dt} = 0 \) as well.
Kinematics

Book’s Notation under the Usual Book Assumptions:

\[ \vec{\omega} \equiv \dot{\theta} \hat{k} \equiv \omega \hat{k} \]

\[ \vec{\alpha} \equiv \frac{d\vec{\omega}}{dt} = \ddot{\theta} \hat{k} \]

\[ \vec{v} \equiv \frac{d\vec{r}}{dt} = r \dot{\theta} \hat{\theta} = \vec{\omega} \times \vec{r} \]

\[ \vec{a} \equiv \frac{d\vec{v}}{dt} = -r \dot{\theta}^2 \hat{r} + r \ddot{\theta} \hat{\theta} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r} \]
Derivatives in Polar Coordinates

\[ \frac{d\hat{r}}{dt} = \dot{\theta} \hat{\theta} \]

\[ \frac{d\hat{\theta}}{dt} = -\dot{\theta} \hat{r} \]

\[ \frac{d\vec{r}}{dt} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \]

\[ \frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta} \]