Exam 2

You May Use:

\[
\frac{dc}{dt} = \dot{\theta} \hat{\theta}
\]

\[
\frac{d\dot{\theta}}{dt} = -\dot{\theta} \ddot{\theta}
\]

\[
\frac{d\ddot{\theta}}{dt} = r \ddot{\theta} + r \dddot{\theta}
\]

\[
\frac{d^2 r}{dt^2} = (\ddot{r} - r \dddot{\theta}^2) \dddot{\theta} + (2 \dddot{\theta} \dddot{\theta} + r \dddot{\theta}) \dot{\theta}
\]

| f(x) - P_n(x, a) | \leq B_{n+1} \frac{|x - a|^{n+1}}{(n + 1)!}.

Under the books assumptions: \( \dddot{\theta} = 0 \), \( \dddot{\theta} = 0 \), and \( \frac{d\dddot{\theta}}{dt} = 0 \):

\[
\vec{\omega} = \dot{\vec{\theta}} \equiv \omega \vec{k}
\]

\[
\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\omega} \vec{k}
\]

\[
\vec{\nu} = \frac{d\vec{\alpha}}{dt} = r \dddot{\theta} = \vec{\omega} \times \dddot{\theta}
\]

\[
\vec{a} = \frac{d\vec{\nu}}{dt} = -r \dddot{\theta} \dddot{\theta} + r \dddot{\theta} \dot{\theta} = \vec{\omega} \times \vec{\nu} + \vec{\alpha} \times \dddot{\theta}
\]
1. Find the real and imaginary parts of $\frac{1}{2x+3}$ with $z = x + iy$.

\[ w = \frac{1}{2z + 3} = \frac{1}{2(x+iy) + 3} \]

\[ = \frac{1}{(2x+3) + i(2y)} \]

\[ = \frac{(2x+3) - i(2y)}{(2x+3)^2 + (2y)^2} \]

So

\[ \text{Re}(w) = \frac{2x+3}{(2x+3)^2 + 4y^2} \]

\[ \text{Im}(w) = \frac{-2y}{(2x+3)^2 + 4y^2} \]
2. Find two distinct vectors of length 1 which are simultaneously perpendicular to line through $(1, 2, 1)$ and $(2, -2, 0)$ and the line through $(2, 1, 3)$ and $(-1, 4, -2)$.

We the lines share their directions with 
\[ \vec{u} = (1, 2, 1) - (2, -2, 0) = ( -1, 4, 1 ) \]
& \[ \vec{w} = (2, 1, 3) - (-1, 4, -2) = (3, 3, 5) \]

hence are both perpendicular to
\[ \vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & 1 \\ 3 & -3 & 5 \end{vmatrix} = 23\hat{i} + 8\hat{j} - 9\hat{k}. \]

Letting $C = |\vec{u} \times \vec{w}| = \sqrt{23^2 + 8^2 + 9^2}$, our two vectors are

\[ \vec{u}_1 = \frac{23}{C} \hat{i} + \frac{8}{C} \hat{j} - \frac{9}{C} \hat{k} \]

\[ \vec{u}_2 = -\vec{u}_1. \]
3. Find the distance between the point \( P = (-25, -13, 8) \) and the plane with equation \( 3x + y - z = 3 \).

(See Example 54 in book)

(Method 2 observes that...)

We need \( \mathbf{e} \) in

\[
(2,5,13,6)
\]

\[
(-26,13,6)
\]

Since

\[ 3 \cdot 1 + 0 - 0 = 3 \]

Where

\[
\mathbf{n} = \frac{(3, 1, -1)}{\sqrt{3^2 + 1^2 + 1^2}}
\]

Since \( \mathbf{n} \cdot \mathbf{n} = 0 \) is our plane with \( \mathbf{n} \) its normal.

So

\[
\mathbf{e} = \left| \frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right| = \left| \frac{(-26, -13, 6) \cdot (25, 13, 8)}{\mathbf{u} \cdot \mathbf{n}} \right|
\]

\[
= \frac{94}{\mathbf{u} \cdot \mathbf{n}} = a \mathbf{u} \mathbf{n}
\]
4. Determine whether each of the following force fields are conservative. If it is, then find a potential function. Justify your answers.

**Method 1**

If \( V \) exists, then \( \nabla V = -\int \mathbf{F} \cdot d\mathbf{r} \) with \( \mathbf{F} = (y \sin(\alpha^2), y) \).

\[
\int \mathbf{F} \cdot d\mathbf{r} = \int_0^x (0, y)(t, \sin(\alpha^2) + t) \cdot (0, 1) dt
\]

\[
= \int_0^y t dt = \frac{-y^2}{2}, \quad \text{but} \quad \frac{\partial}{\partial x} \left( \frac{y^2}{2} \right) = 0 \neq y \sin(\alpha^2)
\]

So \( V \) does not exist.

**Method 2**

If \( V \) exists, then \( \frac{\partial^2 Q}{\partial x \partial y} = y \sin(\alpha^2) \) and \( \frac{\partial^2 Q}{\partial y \partial x} = 0 \).

\[
-\frac{\partial}{\partial x} \left( xy \sin(\alpha^2) + h(y) \right) - \frac{\partial}{\partial y} \left( \frac{y^2}{2} + Q(x) \right) = \frac{y^2}{2} + Q(x)
\]

Well, from this

\[
-\frac{\partial}{\partial x} = y \sin(\alpha^2) = \frac{dg}{dx}(x), \quad \text{A contradiction since} \quad \frac{dg}{dx}(x) \text{ is a function of } x.
\]

So \( V \) does not exist.
Method 1

If $V$ exists, then $V = - \int_{y}^{z} F \cdot d\vec{r}$

with

\[
\begin{align*}
&\{(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z) \to (0,0,0)\} \\
&\{(x,0,0) \to (x,y,0) \to (0,0,0)\}
\end{align*}
\]

(this path has 3 pieces)

\[V = -\int_{0}^{x} (0,1,0) \cdot (1,0,0) \, dt - \int_{0}^{y} (x,2,0) \cdot (0,1,0) \, dt - \int_{0}^{z} (y,t^2 + \frac{2}{3}x^3,0) \cdot (0,0,1) \, dt\]

\[= -0 - \int_{0}^{y} t^2 \, dt - \int_{0}^{z} xy \, dt = -\frac{y^3}{3} - xyz \quad \&
\]

\[\nabla V = (xy, x^2 + \frac{2}{3}x^3, xy) \quad \text{hence} \quad V = -\frac{y^3}{3} - xyz
\]

Method 2

If $V$ exists:

\[\frac{\partial q}{\partial x} = -y - \frac{2}{3}y^3, \quad \frac{\partial q}{\partial y} = -x + \frac{1}{3}x^3 + f_2(x, y), \quad \frac{\partial q}{\partial z} = -xy\]

\[a = -y^2z + (f_3(x, y)) \quad a = -x^2y - \frac{y^3}{3} + f_2(x, y) \quad a = -xy^2 + f_3(x, y)
\]

We have

\[a = \frac{-xy^2 - \frac{y^3}{3}}{3} = V
\]

Our needed potential,
5. (a) What is the Taylor series of $\sin(x)$ around the point $x = 0$?

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

(b) What is the Taylor series of $\cos(x)$ around the point $x = 0$?

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(c) What is the Taylor series of $e^x$ around the point $x = 0$?

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
(d) Using the series from (a)-(c), justify that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

\[
\frac{C}{i\theta} \overset{by\ (a)}{=} \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{2k!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{2k+1!} =
\]

\[
= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} =
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} =
\]

\[
= (\cos(\theta) + i\sin(\theta)) \overset{by\ \text{(a)\ and\ (b)}}{=} \text{as needed.}
\]
6. Let \( \mathbf{u}(3) = (1, -1, 2), \mathbf{v}(3) = (3, 0, -1), \ \frac{d\mathbf{u}}{dt}(3) = (1, 2, 0), \ \frac{d\mathbf{v}}{dt}(3) = (0, -1, 2) \)
and \( \nabla f(1, -1, 2) = (2, 5, 3) \).

(a) Compute \( \frac{d}{dt}(f(\mathbf{u}(t))) \) at \( t = 3 \).

\[
\frac{d}{dt} f(\mathbf{u}(t)) \bigg|_{t=3} = \nabla f(\mathbf{u}(t)) \cdot \frac{d\mathbf{u}}{dt}(t) \bigg|_{t=3}
\]

\[
= (2, 5, 3) \cdot (1, 2, 0) = 12
\]

(b) Compute \( \frac{d}{dt} (\mathbf{u} \cdot \mathbf{v}) \) at \( t = 3 \).

\[
\frac{d}{dt} (\mathbf{u} \cdot \mathbf{v}) \bigg|_{t=3} = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \bigg|_{t=3}
\]

\[
= (1, 2, 0) \cdot (3, 0, -1) + (1, 1, 2) \cdot (0, -1, 2)
\]

\[
= 3 + 5 = 8
\]
(c) Compute \( \frac{d}{dt} (\vec{u} \times \vec{v}) \) at \( t = 3 \).

\[
\frac{d}{dt} (\vec{u} \times \vec{v}) \bigg|_{t=3} = \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt}
\]

\[
= (1, 2, 0) \times (3, 0, -1) + (1, -1, 2) \times (0, 1, 2)
\]

\[
= \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 0 \\
3 & 0 & -1
\end{vmatrix} + \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & -1 & 2 \\
0 & 1 & 2
\end{vmatrix}
\]

\[
= (-2, 1, -6) + (0, -2, -1)
\]

\[
= (-2, -1, -7)
\]
7. A particle travels around a circle with constant speed \( \frac{7 \text{ meters}}{\text{sec}} \) starting at \( 2\pi \) meters. (In other words, \( \dot{\theta} \cdot \frac{dx}{dt} = 7 \). The radius of the circle changes with time as \( r(t) = \frac{2}{1+t} \) meters with \( t \geq 0 \).

(a) Express \( \vec{r}(t) \) in Polar coordinates.

\[
\vec{r}(t) = \hat{r} = \hat{\theta} \cdot (\hat{r} \dot{\hat{r}} + \mathbf{k} \dot{\theta}), \quad r \dot{\theta} = \frac{7}{1+t} \hat{\theta}
\]

so \( \dot{\theta} = \left( \frac{1+t}{2} \right) \frac{7}{1+t} = \frac{7}{2} \left( 1 + \frac{t}{2} \right) \) + 7

\( \theta(0) = 0 \) since we start at \( 2\pi \) so \( c = 0 \)

& \[
\vec{r}(t) = r(t) \hat{r}(\theta(t)) = \frac{7}{1+t} \left( \frac{7}{2} \left( 1 + \frac{t}{2} \right) \right) \mathbf{k}
\]

which is \( \left( \frac{7}{1+t}, \frac{7}{2} \left( 1 + \frac{t}{2} \right) \right) \mathbf{p} \)

(b) Express \( \vec{r}(t) \) in Cartesian coordinates.

\[
\vec{\dot{r}}(t) = \frac{d}{dt} \left( r(t) \hat{r}(\theta(t)) \right) = r(t) \left( \cos(\theta(t)) \hat{i} + \sin(\theta(t)) \hat{j} \right)
\]

\[
= \frac{7}{1+t} \cos \left( \frac{7}{2} \left( 1 + \frac{t}{2} \right) \right) \hat{i} + \frac{7}{1+t} \sin \left( \frac{7}{2} \left( 1 + \frac{t}{2} \right) \right) \hat{j}
\]
8. Suppose you have a function \( f(x) \) such that \( f(x) \)'s third Taylor polynomial at \( x = 1 \) is \( P_3(x) = 1 - (1/2)(x-1) + (x-1)^2 + (2/3)(x-1)^3 \), and assume that all of \( f(x) \)'s derivatives are bounded by 5 on the interval \((0, 2)\) (in other words \( \left| \frac{d^4}{dx^4} f(x) \right| < 5 \) for \( 0 \leq x \leq 2 \)).

(a) Given the above data, approximate \( f(1.5) \).

\[
f(1.5) = f\left(\frac{3}{2}\right) = 1 - \frac{1}{2} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \frac{2}{3} \left(\frac{1}{2}\right)^3 \\
= 1 + \frac{1}{12} = \frac{13}{12}
\]

(b) Bound the difference \( |f(1.5) - P_3(1.5)| \) using the above data, and justify your answer.

\[
|f\left(\frac{3}{2}\right) - P_3\left(\frac{3}{2}\right)| \leq \frac{5}{4!} \leq \frac{5}{24} \leq \frac{5}{27.3} = \frac{1}{64}
\]

\[
\max_{\left[1, \frac{3}{2}\right]} |f''(x)| \leq 5
\]

(c) Given the above data, can you determine \( f(x) \)'s second derivative at \( x = 1? \) If so find it, if not why.

\[
\text{Well } P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!} \\
\text{& so } \frac{f''(1)}{2!} = 1 \text{ & } f'''(1) = 2
\]