LECTURE OUTLINE
Complex Numbers

Professor Leibon

Math 15

Nov. 3, 2004
Goals

Introduce Complex Numbers
Geometry of Complex Numbers
Polar Form

We can write a complex number \( z \) in **cartesian** or **polar** form

\[
z = x + yi = r(\cos(\theta) + i \sin(\theta)) \equiv re^{i\theta},
\]

and we say \( \text{arg}(z) = \theta \) is \( z \)'s argument, \( |z| = r \) is \( z \)'s norm, \( x = \text{Re}(z) \) is \( z \)'s real part, and \( y = \text{Im}(z) \) is \( z \)'s imaginary part,

Find \( z = \frac{5\sqrt{3}}{2} + i\frac{5}{2} \) norm, argument, real part, imaginary part, and express \( z \) in polar form.
Addition

Let $z = x_1 + y_1 i$ and $w = x_2 + y_2 i$. Then $z + w = (x_1 + x_2) + (y_1 + y_2)i$, is simply vector addition.

Add $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ to $w = 1 + i$. 
Multiplication

Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$. Then

$$zw = (r_1 r_2) e^{i(\theta_1 + \theta_2)}.$$  

Check that if we express this in cartesian form that we are simply "FOILing".

Square $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ and describe the norm and argument of the square.

Multiply $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$ and $w = 1 + i$. 
Powers

Let \( z = re^{i\theta} \) and note \( z^n = r^n e^{in\theta} \). This tells us how to take roots, namely

\[
(z)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta + k2\pi}{n}\right)}
\]

for any \( k \) from 0 to \( n - 1 \).

Find the square, cube, 4th and 5th roots of 1. Graph them. (These are called the roots of unity.)
Let $z = x + yi = re^{i\theta}$ be a non-zero complex number, then $z$’s inverse is

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2} = \frac{1}{r}e^{-i\theta}$$

and we say $\bar{z} = x - yi$ is $z$’s complex conjugate, and note $z^{-1} = \frac{\bar{z}}{|z|^2}$.

Find the inverse of $z = \frac{5\sqrt{3}}{2} + i\frac{5}{2}$ and express it is polar and cartesian form.
Examples

1. Express $1 + \frac{2}{1-i}$ in the form $a + bi$.

2. Let $z = x + yi$ and find the real and imaginary parts of

$$\frac{z + 1}{3z - 2}$$

3. Simplify $(1 - i)^5$. 
