LECTURE OUTLINE
Driven Harmonic Motion

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Math 15
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Goal

Driven Harmonic Motion

IHOLDES
Last Time

Let \( \lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4c}}{2} \). We have used the power series and factoring methods to find that \( e^{\lambda_{\pm}z} \) are solutions to

\[
\Delta f = D^2 f + \gamma D f + cf = (D - \lambda_+)(D - \lambda_-)f = 0,
\]

and we get our needed pair of solutions provided \( \gamma^2 - 4c \neq 0 \). (Notation: \( \gamma = \frac{\beta}{m} > 0 \) and \( c = \frac{k}{m} = \omega^2 > 0 \).)

We are interested in solving the initial value problem \( f(0) = A \) and \( Df(0) = B \) for real constant \( A \) and \( B \). Since we are good at solving linear equations, we simply use our above solutions to find two distinct real solutions.
The $\gamma^2 - 4c \neq 0$ Solutions

Case 1: If $\gamma^2 > 4c$, then $e^{\lambda \pm t}$ is real and our real solutions are in the form $f(t) = c_0 e^{\lambda t} + d_0 e^{\lambda - t}$.

Case 2: $\gamma^2 < 4c$, we have the two real solutions

$$\frac{1}{2} (e^{\lambda t} + e^{\lambda - t}) = Re(e^{\lambda t}) = e^{\frac{-\gamma}{2}} \cos(t \frac{\sqrt{4c - \gamma^2}}{2})$$

$$\frac{-i}{2} (e^{\lambda t} - e^{\lambda - t}) = Im(e^{\lambda t}) = e^{\frac{-\gamma}{2}} \sin(t \frac{\sqrt{4c - \gamma^2}}{2}),$$

and all our real solutions are in the form

$$f(t) = c_0 e^{\frac{-\gamma}{2}} \cos(t \frac{\sqrt{4c - \gamma^2}}{2}) + d_0 e^{\frac{-\gamma}{2}} \cos(t \frac{\sqrt{4c - \gamma^2}}{2}).$$
Examples

Solve

\[ D^2 f + 3Df + 4f = 0 \]

for \( f(0) = 1 \) and \( Df(0) = 2 \).

Solve

\[ D^2 f + Df + 4f = 0 \]

for \( f(0) = 0 \) and \( Df(0) = 1 \).
Driven Motion

We have been exploring how to solve

$$\Delta f = D^2 f + p(z)Df + q(z)f = 0.$$ 

Now we try and solve the inhomogenous version of this equation

$$\Delta f = G$$

for some fixed function $G$. The key is the following:

**Main Theorem:** Suppose $\Delta f_p = G$, then every solution to $\Delta f = G$ is in the form $f_p + f_h$ where $f_h$ is a solution to $\Delta f_h = 0$. 