LECTURE OUTLINE

Solving Differential Equations By Factoring

Professor Leibon

Math 15

Nov. 15, 2004
Goal

Review: Power Series Method
Differential Operators
Factoring Differential Equations
Summary of Last Night

**Theorem:** Given function \( p(z) \) and \( q(z) \) that can be described by power series that converges for \( |z - z_0| \leq R \) and the equation

\[
\frac{d^2 f}{d z^2} + p(z) \frac{df}{dz} + q(z) f = 0,
\]

the *power series method* will always produce a solution to this equation that can also described as a power series that converges for \( |z - z_0| \leq R \) subject any choice of *initial conditions* \( f(z_0) = A \) and \( \frac{df}{dx}(z_0) = B \).

Furthermore any solution to such an equation is uniquely determined by its initial conditions \( f(z_0) = A \) and \( \frac{df}{dx}(z_0) = B \).
Let $D = \frac{d}{dz}$ (the act of taking a derivative). $D^n$ means perform this action $n$ times. Then our previous equation can be written as

$$D^2 f + p(z)Df + q(z)f = 0.$$  

As such we are viewing our differential equation as something we do. $\Delta = D^2 f + p(z)Df + q(z)f$ an example of an linear differential operator.
Example 4

Using the $\Delta f = D^2 f + \omega^2$, solve the equation

$$\Delta f = 0$$

$$f(0) = A$$

$$Df(0) = B$$
Example 5

Let $\Delta = D^2 f + \omega^2$. Factor and find two solutions to

$$\Delta f = 0.$$
Theorem: If $\Delta f = D^2 f + p(z) D f + q(z) f$ and $c$ and $d$ are any constants, then
$\Delta (cg + dh) = c \Delta g + d \Delta h$.

Corollary: If $\Delta f = D^2 f + p(z) D f + q(z) f$ and $g$ and $h$ solve the equation $\Delta f = 0$, then $cg + dh$ also solves this equation.
Example 6

Let $\Delta = D^2 + \omega^2$. Solve

$$\Delta f = 0$$

$$f(0) = A$$

$$\frac{df}{dz}(0) = B$$

in two ways.
Key Fact: To solve the equation $\Delta f = 0$ for any initial conditions $f(0) = A$ and $Df(0) = B$ we can simply find 2 solutions $g(z)$ and $h(z)$ such that for some choice of $c$ and $d$ we can solve $cf(0) + dh(0) = A$ and $cDf(0) + dDh(0) = B$. 
Example 7: Damped Harmonic Motion

Try to find all solutions to

$$\frac{d^2 f}{dz^2} + \gamma \frac{df}{dz} + cf = 0$$

with $\gamma > 0$ and $c > 0$. 
Critically Damped Harmonic Motion

Use the power series method to find a solution to

\[ \frac{d^2 f}{dz^2} + 2\omega \frac{df}{dz} + \omega^2 f = 0 \]

\[ f(0) = 0 \]

\[ \frac{df}{dz}(0) = 1. \]

Using the key fact, find the solution to this equation.
Example 7 (Bessel’s Equation)

Try to solve

\[ z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f = 0 \]

\[ f(0) = A \]

\[ \frac{df}{dz}(0) = B \]

What happens?